(B) The incompressible Navier-Stokes Equation

See also Chapter 2 from Frisch 1995.

Velocity-pressure formulation

$$\begin{aligned} \partial_t \mathbf{v} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} &= -\boldsymbol{\nabla} p + \nu \triangle \mathbf{v} \\ \boldsymbol{\nabla} \cdot \mathbf{v} &= 0 \\ \mathbf{v}|_{\partial\Lambda} &= \mathbf{0} \end{aligned}$$

Here $D_t = \partial_t + \mathbf{v} \cdot \boldsymbol{\nabla}$ is material or convective derivative; ν is kinematic viscosity.

Pressure and Poisson equation:

$$\begin{aligned} \boldsymbol{\nabla} \cdot [(\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v}] &= \partial_i((v_j \partial_j) v_i) = (\partial_i v_j)(\partial_j v_i) = (\boldsymbol{\nabla}\mathbf{v})^\top : \boldsymbol{\nabla}\mathbf{v} \\ -\triangle p &= (\boldsymbol{\nabla}\mathbf{v})^\top : \boldsymbol{\nabla}\mathbf{v} \end{aligned}$$

N-S on $\partial\Lambda$:

$$0 = -\boldsymbol{\nabla} p + \nu \frac{\partial^2 \mathbf{v}}{\partial n^2}$$

Neumann b.c.

$$\frac{\partial p}{\partial n} = \nu n_i \frac{\partial^2 v_i}{\partial n^2}$$

Vorticity-velocity formulation

$$\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{v} = \text{ vorticity}$$

identity: $\mathbf{v} \times \boldsymbol{\omega} = \boldsymbol{\nabla}(\frac{1}{2}v^2) - (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v}$

Proof:

$$\begin{aligned} (\mathbf{v} \times \boldsymbol{\omega})_i &= \epsilon_{ijk}\epsilon_{lmk} v_j \partial_l v_m \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) v_j \partial_l v_m \\ &= v_j \partial_i v_j - (v_j \partial_j) v_i \end{aligned}$$

$$\therefore \text{ N-S } \Longrightarrow \partial_t \mathbf{v} = \mathbf{v} \times \boldsymbol{\omega} - \boldsymbol{\nabla} p' + \nu \triangle \mathbf{v}$$

$$\begin{aligned} p' &= p + \frac{1}{2}v^2 \\ \mathbf{v}\times\boldsymbol{\omega} &= \text{vortex force} \end{aligned}$$

<u>identity:</u> $\boldsymbol{\nabla}\times(\mathbf{v}\times\boldsymbol{\omega}) = -(\mathbf{v}\cdot\boldsymbol{\nabla})\boldsymbol{\omega} + (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\mathbf{v}$

Proof: Use $\boldsymbol{\nabla}\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\boldsymbol{\nabla}\cdot\mathbf{b}) - \mathbf{b}(\boldsymbol{\nabla}\cdot\mathbf{a}) + (\mathbf{b}\cdot\boldsymbol{\nabla})\mathbf{a} - (\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{b}$ and incompressibility. Thus $\boldsymbol{\nabla}\times(NS) \Longrightarrow$

$$\partial_t\boldsymbol{\omega} + (\mathbf{v}\cdot\boldsymbol{\nabla})\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\mathbf{v} + \nu\triangle\boldsymbol{\omega}$$

Here $(\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\mathbf{v}$ is the *vortex-stretching term.*

<u>Poisson equation for velocity:</u> $-\triangle\mathbf{v} = \boldsymbol{\nabla}\times\boldsymbol{\omega}$

Proof: use $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{a}) = -\triangle\mathbf{a} + \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{a})$ to take curl of $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\mathbf{v}$

<u>Biot-Savart formula:</u> For $D(\mathbf{r},\mathbf{r}')$ the Green's function of the Laplacian with Dirichlet b.c.

$$\begin{aligned} \mathbf{v}(\mathbf{r}) &= \int_V d^d\mathbf{r}'\ D(\mathbf{r},\mathbf{r}')\,(\boldsymbol{\nabla}_{r'}\times\boldsymbol{\omega})(\mathbf{r}') \\ &= \int_V d^d\mathbf{r}'\ \mathbf{K}(\mathbf{r},\mathbf{r}')\times\boldsymbol{\omega}(\mathbf{r}') + \int_{\partial V} dS'\,D(\mathbf{r},\mathbf{r}')\,\mathbf{n}'\times\boldsymbol{\omega}(\mathbf{r}'), \end{aligned}$$

with $\mathbf{K}(\mathbf{r},\mathbf{r}') = -\boldsymbol{\nabla}_{r'}D(\mathbf{r},\mathbf{r}')$. This is called the "Biot-Savart formula" because of an analogy with magnetostatics:

$$\begin{aligned} \boldsymbol{\nabla}\times\mathbf{B} &= 4\pi\mathbf{J} \\ \mathbf{B} &\longleftrightarrow \mathbf{v} \\ \mathbf{J} &\longleftrightarrow \boldsymbol{\omega} \end{aligned}$$

<u>Symmetries of NS</u>

A group G of transformations of $\mathbf{v}(\mathbf{r},t)$ is <u>symmetry group</u> of NS if and only if

$\forall g\in G$, $\mathbf{v}$ a NS solution. $\Longrightarrow g\mathbf{v}$ a NS solution.

$$\begin{aligned} \text{Space-translations}: g_{\mathbf{a}}^{space}\mathbf{v}(\mathbf{r},t) &= \mathbf{v}(\mathbf{r}-\mathbf{a},t), \quad \mathbf{a}\in\mathbb{R}^d, \\ \text{Time-translations}: g_{\tau}^{time}\mathbf{v}(\mathbf{r},t) &= \mathbf{v}(\mathbf{r},t-\tau), \quad \tau\in\mathbb{R}, \\ \text{Galilean transformation}: g_{\mathbf{u}}^{Gal}\mathbf{v}(\mathbf{r},t) &= \mathbf{v}(\mathbf{r}-\mathbf{u}t,t)+\mathbf{u}, \quad \mathbf{u}\in\mathbb{R}^d \end{aligned}$$

$$\begin{aligned}
\text{Space-reflection (parity)}: \ P\mathbf{v}(\mathbf{r},t) &= -\mathbf{v}(-\mathbf{r},t) \\
\Lambda = \mathbb{R}^d \Longrightarrow \text{Space-rotations}: g_R^{rot}\mathbf{v}(\mathbf{r},t) &= R\mathbf{v}(R^{-1}\mathbf{r},t), \quad R \in SO(d) \\
\nu = 0 \Longrightarrow \text{Scale-invariance}: \ g_\lambda^{scale,h}\mathbf{v}(\mathbf{r},t) &= \lambda^h \mathbf{v}(\lambda^{-1}\mathbf{r}, \lambda^{h-1}t), \quad \lambda \in \mathbb{R}^+
\end{aligned}$$

Note that space domain $\tilde{\Lambda} = \lambda\Lambda$ and time interval $\tilde{T} = \lambda^{1-h}T$. For $\nu > 0$, scale-invariance holds only with $h = -1$

$$g_\lambda^{scale}\mathbf{v}(\mathbf{r},t) = \lambda^{-1}\mathbf{v}(\lambda^{-1}\mathbf{r}, \lambda^{-2}t)$$

Proof for Galilean transformation: Set $\tilde{\mathbf{v}}(\mathbf{r},t) = \mathbf{v}(\mathbf{r}-\mathbf{u}t,t) + \mathbf{u}$

$$\begin{aligned}
\partial_t \tilde{\mathbf{v}}(\mathbf{r},t) &= \partial_t \mathbf{v}(\mathbf{r}-\mathbf{u}t,t) - \underbrace{(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{v}(\mathbf{r}-\mathbf{u}t,t)} \\
(\tilde{\mathbf{v}}(\mathbf{r},t)\cdot\boldsymbol{\nabla})\tilde{\mathbf{v}}(\mathbf{r},t) &= [\mathbf{v}(\mathbf{r}-\mathbf{u}t,t)\underbrace{+\mathbf{u}}]\cdot\boldsymbol{\nabla}\mathbf{v}(\mathbf{r}-\mathbf{u}t,t)
\end{aligned}$$

$\underbrace{\ \ }$ terms cancel!

Proof for scale-invariance: Set $\tilde{\mathbf{v}}(\mathbf{r},t) = \lambda^h\mathbf{v}(\lambda^{-1}\mathbf{r}, \lambda^{h-1}t)$

$$\begin{aligned}
\partial_t \tilde{\mathbf{v}}(\mathbf{r},t) &= \lambda^{2h-1}\mathbf{v}(\lambda^{-1}\mathbf{r}, \lambda^{h-1}t) \\
(\tilde{\mathbf{v}}(\mathbf{r},t)\cdot\boldsymbol{\nabla})\tilde{\mathbf{v}}(\mathbf{r},t) &= \lambda^{2h-1}\mathbf{v}(\lambda^{-1}\mathbf{r}, \lambda^{h-1}t)\cdot\boldsymbol{\nabla}\mathbf{v}(\lambda^{-1}\mathbf{r}, \lambda^{h-1}t) \\
\boldsymbol{\nabla}\tilde{p}(\mathbf{r},t) &= \lambda^{2h-1}\boldsymbol{\nabla}p(\lambda^{-1}\mathbf{r}, \lambda^{h-1}t) \quad \text{(Why?)} \\
\nu\nabla\tilde{\mathbf{v}}(\mathbf{r},t) &= \lambda^{h-2}\nu\nabla\mathbf{v}(\lambda^{-1}\mathbf{r}, \lambda^{h-1}t)
\end{aligned}$$

For $\nu > 0$, $h - 2 = 2h - 1 \Longrightarrow h = -1$

Define the Reynolds number

$$Re = \frac{UL}{\nu}, \quad L = \text{ domain size}, \quad U = \text{ characteristic velocity}$$

Since $\tilde{L} = \lambda L$, $\tilde{U} = \lambda^h U$

$$\tilde{Re} = \lambda^{h+1}Re = Re, \quad \text{if} \quad h = -1$$

Principle of hydrodynamic similarity: Two flows with the same geometry but different scale are essentially identical if the Reynolds numbers are the same.

Non-dimensionalization:

$$\begin{aligned}\hat{\mathbf{v}} &= \mathbf{v}/U \\ \hat{\mathbf{r}} &= \mathbf{r}/L \\ \hat{t} &= (U/L)t \;=\; t/T\end{aligned}$$

$\Longrightarrow$

$$\partial_{\hat{t}}\hat{\mathbf{v}} + (\hat{\mathbf{v}}\cdot\hat{\boldsymbol{\nabla}})\hat{\mathbf{v}} = -\hat{\boldsymbol{\nabla}}\hat{p} + \frac{1}{Re}\hat{\triangle}\hat{\mathbf{v}}$$

An important consequence of the similarity principle is the fact that the rescaled energy dissipation $D = \varepsilon(t)/(U^3/M)$ in decaying grid turbulence with inflow velocity U and mesh size M can be a function only of $Re_M = UM/\nu$, dimensionless time $\hat{t} = Ut/M$, and scale-independent geometric properties of the grid. See homework!

The hydrodynamic similarity principle is a special case of the Buckingham Π-Theorem, which implies among other things that, if there are n quantities Q_i, $i = 1, .., n$ and k independent physical dimensions, then there are exactly $p = n-k$ independent dimensionless number groups $\Pi_j = Q_1^{a_{1j}} Q_2^{a_{2j}} \cdots Q_n^{a_{nj}}$, $j = 1, ..., p$ for rational numbers a_{ij}, $i = 1, ..., n$, $j = 1, ..., p$. See:

> Buckingham, E. "On physically similar systems; illustrations of the use of dimensional equations," Physical Review. **4** 345–376 (1914)

Thus, from the three quantities U, L, ν with two independent dimensions of (*length*) and (*time*), one can construct a single dimensionless group, which may be the Reynolds number $Re = UL/\nu$ or, alternatively, some rational power of it, such as $\sqrt{Re}$.

Incompressible fluctuating hydrodynamics: We mention here that effects of thermal noise on incompressible fluids can be described by a nonlinear Langevin equation/stochastic PDE

$$\partial_t\mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = -\boldsymbol{\nabla}p + \nu\triangle\mathbf{v} + \mathbf{f}$$

where p is chosen to enforce $\boldsymbol{\nabla}\cdot\mathbf{v} = 0$ and $f_i(\mathbf{x}, t)$ is a Gaussian space-time white-noise random process with mean zero and covariance prescribed by a fluctuation-dissipation relation:

$$\langle f_i(\mathbf{x},t) f_j(\mathbf{x}',t')\rangle = \frac{2\nu k_B T}{\rho}\delta_{ij}\triangle_x \delta^d(\mathbf{x}-\mathbf{x}')\delta(t-t')$$

where $k_B \doteq 1.3806 \times 10^{-23}\ J/K$ is Boltzmann's constant. See the careful derivation in the paper of Donev et al. (2014) cited at the end of the last section.

If the same rescaling is performed as for the deterministic equation, one obtains

$$\partial_{\hat{t}}\hat{\mathbf{v}} + (\hat{\mathbf{v}}\cdot\hat{\boldsymbol{\nabla}})\hat{\mathbf{v}} = -\hat{\boldsymbol{\nabla}}\hat{p} + \frac{1}{Re}\hat{\triangle}\hat{\mathbf{v}} + \hat{\mathbf{f}}$$

where $\hat{\mathbf{f}} = L\mathbf{f}/U^2$ has covariance

$$\langle \hat{f}_i(\hat{\mathbf{x}},\hat{t}) \hat{f}_j(\hat{\mathbf{x}}',\hat{t}')\rangle = \frac{2}{Re}\theta_L \delta_{ij}\triangle_{\hat{x}} \delta^d(\hat{\mathbf{x}}-\hat{\mathbf{x}}')\delta(\hat{t}-\hat{t}')$$

with parameter $\theta_L = k_B T/(\rho U^2 L^d)$ that compares thermal fluctuation energy to total hydrodynamic energy of density $\sim \rho U^2$ in a volume of order L^d. An important conclusion is that thermal noise introduces a new dimensionless constant θ_L and thus *breaks hydrodynamic similarity*. However, if U and L are chosen to be large velocity and length-scales corresponding to the largest turbulent eddies, then we shall see that θ_L is incredibly tiny, and the *direct* effects are small. Even if U and L are chosen to be small velocity and length-scales corresponding to the tiniest turbulent eddies, then θ_L is very small, usually $\epsilon \sim 10^{-9} - 10^{-6}$. Nevertheless, we shall see later that turbulent fluctuations become even tinier at those scales and thus thermal fluctuations dominate.

"Inviscid Invariants" of 3D NS

Momentum:

$$\mathbf{g}(\mathbf{r},t) = \rho\mathbf{v}(\mathbf{r},t) = \text{ momentum density}$$

$$\begin{aligned} \text{NS} \implies \partial_t \mathbf{g} + \boldsymbol{\nabla}\cdot\mathbf{T} &= 0 \\ \mathbf{T} = P\mathbf{I} + \rho\mathbf{v}\mathbf{v}^\top - 2\eta\mathbf{S} &= \text{ stress tensor (spatial momentum flux)} \\ \mathbf{S} = \frac{\boldsymbol{\nabla}\mathbf{v} + (\boldsymbol{\nabla}\mathbf{v})^\top}{2} &= \text{ strain tensor, } (\boldsymbol{\nabla}\mathbf{v})_{ij} = \partial_i v_j \end{aligned}$$

Global conservation:

$$\begin{aligned} \mathbf{P}(t) &= \int_\Lambda \mathbf{g}(\mathbf{r},t) d^d\mathbf{r} = \text{ total momentum} \\ \frac{d\mathbf{P}(t)}{dt} &= 0 \ for \ \Lambda = \mathbb{R}^d \text{ on } \mathbb{T}^d \end{aligned}$$

Local conservation: $B \subset \Lambda$

$$\begin{aligned} \mathbf{P}_B(t) &= \int_B \mathbf{g}(\mathbf{r},t) d^d\mathbf{r} = \text{ momentum in } B \\ \frac{d\mathbf{P}_B(t)}{dt} &= -\int_{\partial B} \mathbf{T} \cdot d\mathbf{A} = \text{ flow of momentum across boundary of } B \end{aligned}$$

<u>Energy:</u>

$$e(\mathbf{r},t) = \frac{1}{2} v^2(\mathbf{r},t) \ \ (k(\mathbf{r},t) = \frac{1}{2}\rho v^2(\mathbf{r},t))$$

$$\underline{\partial_t e + \boldsymbol{\nabla} \cdot \mathbf{J}_E = -\varepsilon_E}$$

with

$$\begin{aligned} \mathbf{J}_E &= (e+p)\mathbf{v} - \nu \boldsymbol{\nabla} e \\ \varepsilon_E &= \nu |\boldsymbol{\nabla} \mathbf{v}|^2 = \nu \sum_{ij} (\partial_i v_j)^2 \ (= \varepsilon) \end{aligned}$$

$$\begin{aligned} E(t) &= \int e(\mathbf{r},t) d^d\mathbf{r}, \ \ \mathcal{E}_E(t) = \int \varepsilon_E(\mathbf{r},t) d^d\mathbf{r} \\ \frac{\partial E}{\partial t} &= -\varepsilon_E, \ \nu \to 0 \Longrightarrow \text{formally } \mathcal{E}_E = 0, \ dE/dt = 0 \end{aligned}$$

<u>Helicity(d = 3) :</u>

$$h(\mathbf{r},t) = \mathbf{v}(\mathbf{r},t) \cdot \boldsymbol{\omega}(\mathbf{r},t)$$

$$\underline{\partial_t h + \boldsymbol{\nabla} \cdot \mathbf{J}_H = -\varepsilon_H}$$

with

$$\begin{aligned} \mathbf{J}_H &= h\mathbf{v} + (p-e)\boldsymbol{\omega} - \nu \boldsymbol{\nabla} h \\ \varepsilon_H &= 2\nu \boldsymbol{\nabla} \mathbf{v} : \boldsymbol{\nabla} \boldsymbol{\omega} = 2\nu \sum_{ij} (\partial_i v_j)(\partial_i \omega_j) \end{aligned}$$

$$H(t) = \int h(\mathbf{r},t)d^3\mathbf{r}, \;\; \mathcal{E}_H(t) = \int \varepsilon_H(\mathbf{r},t)d^3\mathbf{r}$$
$$\frac{dH}{dt}(t) = -\mathcal{E}_H, \;\; \nu \to 0 \Longrightarrow \text{ formally } \mathcal{E}_H = 0, \;\; dH/dt = 0$$

The helicity integral

$$H = \int d^3x \; \mathbf{v}(\mathbf{x},t)\cdot\boldsymbol{\omega}(\mathbf{x},t)$$

has an interesting interpretation in terms of the topology of vortex-lines. For example, for two vortex tubes T_1, T_2 with vorticity fluxes Φ_1, Φ_2 (and no twist!)

$$H = 2lk(T_1,T_2)\Phi_1\Phi_2$$

where $lk(T_1,T_2)$ is the Gauss linking number:

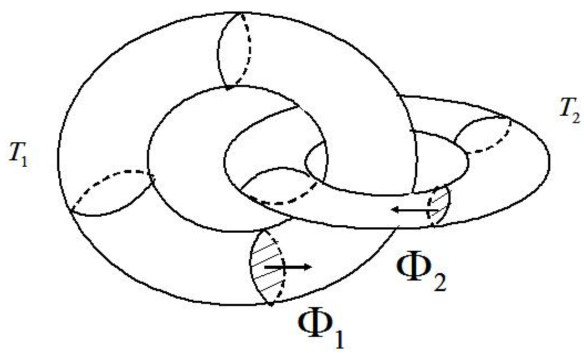

Figure 1. $H = \pm 2\Phi_1\Phi_2$

More generally, the helicity represents the average self-linking of the vortex lines. See:

H.K. Moffatt, "The degree of knottedness of tangled vortex lines," J. Fluid Mech., **106**, 117-129 (1969).

V.I. Arnold & B.A. Khesin, *Topological Methods in Hydrodynamics*, Springer, NY, 1998, Section III.

Proof for energy: $e = \frac{1}{2}v_i^2$

$$\begin{aligned}
\partial_t e &= v_i \dot{v}_i \\
&= v_i[-(\mathbf{v}\cdot\boldsymbol{\nabla})v_i - \partial_i p + \nu \partial_j^2 v_i] \\
&= -(\mathbf{v}\cdot\boldsymbol{\nabla})(\frac{1}{2}v_i^2) - \partial_i(p v_i) + \partial_j(\nu v_i \partial_j v_i) - \nu(\partial_j v_i)^2 \\
&= -\boldsymbol{\nabla}\cdot[(e+p)\mathbf{v}] + \boldsymbol{\nabla}\cdot[\nu\boldsymbol{\nabla}e] - \varepsilon_k
\end{aligned}$$

Proof for helicity: $h = v_i \omega_i$

$$\begin{aligned}
\partial_t h &= v_i \dot{\omega}_i + \dot{v}_i \omega_i \\
&= v_i[-(\mathbf{v}\cdot\boldsymbol{\nabla})\omega_i + (\omega_j \partial_j)v_i + \nu\partial_j^2 \omega_i] \\
&\quad +\omega_i[-(\mathbf{v}\cdot\boldsymbol{\nabla})v_i - \partial_i p + \nu\partial_j^2 v_i] \\
&= -(\mathbf{v}\cdot\boldsymbol{\nabla})(v_i\omega_i) + (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})(\frac{1}{2}v_i^2 - p) \\
&\quad +\nu\partial_j(v_i\partial_j\omega_i + \omega_i\partial_j v_i) - 2\nu\partial_j v_i \partial_j \omega_i \\
&= -\boldsymbol{\nabla}\cdot[h\mathbf{v} + (p-e)\boldsymbol{\omega}] + \boldsymbol{\nabla}\cdot[\nu\boldsymbol{\nabla}h] - \varepsilon_H
\end{aligned}$$

Remark # 1: Set $S_{ij} = \frac{1}{2}(\partial_i v_j + \partial_j v_i)$ and $\Omega_{ij} = \frac{1}{2}(\partial_i v_j - \partial_j v_i) = -\frac{1}{2}\epsilon_{ijk}\omega_k$. Then, $\varepsilon_E = \nu|\boldsymbol{\nabla}\mathbf{v}|^2 = \nu(S^2 + \Omega^2) = \nu(S^2 + \frac{1}{2}\omega^2)$ using $\epsilon_{ijk}\epsilon_{ijl} = 2\delta_{kl}$. From

$$\begin{aligned}
-\triangle p &= (\boldsymbol{\nabla}\mathbf{v})^\top : \boldsymbol{\nabla}\mathbf{v} \\
&= S^2 - \Omega^2 = S^2 - \frac{1}{2}\omega^2
\end{aligned}$$

$$\Longrightarrow \int S^2 d^d\,\mathbf{r} = \frac{1}{2}\int \omega^2 d^d\mathbf{r}$$

$$\begin{aligned}
\therefore \mathcal{E}_E &= 2\nu\int S^2 d^d\mathbf{r} = \nu\int \omega^2 d^d\mathbf{r} = 2\nu\Omega \\
\Omega &= \frac{1}{2}\int \omega^2 d^d\,\mathbf{r} = \text{ enstrophy}
\end{aligned}$$

Note that only $2\nu S^2$ represents the true dissipation, in that it is only this term which appears in the equation for internal energy u and which describes local heating of the fluid.

Remark # 2 (d=2) : $\partial_t\omega + \boldsymbol{\nabla}\cdot[\mathbf{v}\omega] = \nu\triangle\omega$ (no vortex stretching!)

$$\partial_t(\frac{1}{2}\omega^2) + \boldsymbol{\nabla}\cdot\left[\frac{1}{2}\omega^2\mathbf{v} - \nu\boldsymbol{\nabla}(\frac{1}{2}\omega^2)\right] = -\nu|\boldsymbol{\nabla}\omega|^2.$$

Instead there is an extra term $\omega^\top S\omega$ *for* $d=3$!

$$\begin{aligned}\Gamma(t) &= \int \omega(\mathbf{r},t)d^2\mathbf{r} = \text{ total circulation}\\ P(t) &= \frac{1}{2}\int |\boldsymbol{\nabla}\omega(\mathbf{r},t)|^2 d^2\mathbf{r} = \text{ palinstrophy}\end{aligned}$$

$\Longrightarrow$

$$\begin{aligned}\frac{d\Gamma}{dt}(t) &= 0\\ \frac{d\Omega}{dt}(t) &= -2\nu P(t)\\ \therefore \text{formally } \nu\to 0 \implies & \frac{d\Omega}{dt}=0\end{aligned}$$

Remark # 3: Since the mean dissipation

$$\varepsilon = \nu\langle|\boldsymbol{\nabla}\mathbf{v}|^2\rangle,$$

it follows that

$$\langle|\boldsymbol{\nabla}\mathbf{v}|^2\rangle \sim \frac{\varepsilon}{\nu}\sim O(Re)$$

as $Re\to\infty$. Thus, the velocity field must be non-differentiable (in mean-square sense) as $Re\to\infty$. This was described by Onsager (1945) as a "violet catastrophe", or what physicists now call an "ultraviolet (or short-distance) divergence." Also

$$\varepsilon = 2\nu\Omega$$

where $\Omega=\frac{1}{2}\langle\omega^2\rangle$ is the enstrophy. Thus

$$\Omega\sim\frac{\varepsilon}{2\nu}\sim O(Re).$$

Hence, turbulence at high Reynolds number has some efficient mechanism of generation of large amounts of enstrophy. This mechanism was identified by G.I. Taylor (1917, 1937) as vortex-stretching, a vorticity-magnification due to turbulent vortex-line growth (see below!)

This cannot occur in 2D! In 2D:

$$\frac{d\Omega}{dt} = -2\nu P(t)\leq 0 \Longrightarrow \Omega(t)\leq\Omega(t_0)$$

Thus,

$$\begin{aligned} \varepsilon(t) &= 2\nu\Omega(t) \leq 2\nu\Omega(t_0) = O(\frac{1}{Re}) \\ \text{and} \quad & \lim_{\nu\to 0}\varepsilon(t) = 0 \text{ in 2D!} \end{aligned}$$

Kelvin Circulation Theorem

Circulation around loop C:

$$\Gamma_C = \oint_C \mathbf{v}\cdot d\mathbf{x} = \int_S \boldsymbol{\omega}\cdot d\mathbf{A} \quad \text{(Stokes theorem)}$$

where S is any surface that spans the loop C:

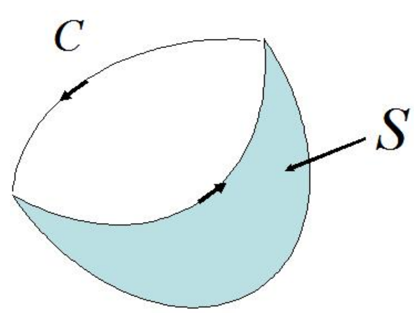

Figure 2. Vorticity flux through surface S bounded by C.

Lagrangian map

$$\begin{aligned} \frac{d\mathbf{X}}{dt}(a,t) &= \mathbf{v}(\mathbf{X}(a,t),t) \\ \mathbf{X}(a,0) &= a \end{aligned}$$

$$\Gamma_C(t) \quad = \quad \oint_{C(t)} \mathbf{v}(t) \cdot d\mathbf{x}$$

Kelvin Theorem

$$\frac{d}{dt}\Gamma_C(t) = \nu \oint_{C(t)} \triangle \mathbf{v}(t) \cdot d\mathbf{x}$$
$$\nu \to 0 \Longrightarrow \frac{d}{dt}\Gamma_C(t) = 0$$

Note: Kelvin's Theorem is equivalent to the Navier-Stokes equations!

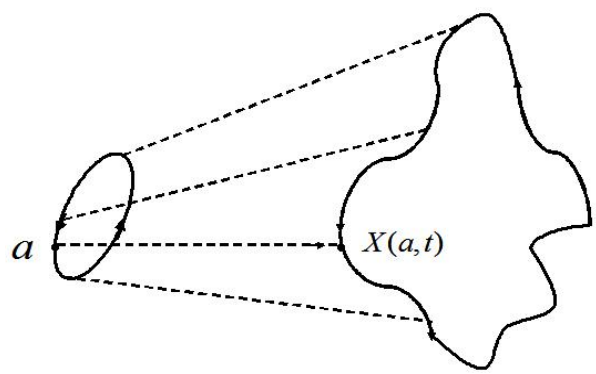

Figure 3. Evolution of a material loop $C(t)$

Proof of Kelvin Theorem:

$$\begin{aligned}
\Gamma_{C(t)} &= \oint_{C(t)} \mathbf{v}(t) \cdot d\mathbf{l} \\
&= \oint_{C(0)} \mathbf{v}(\mathbf{X}(t), t) \cdot d\mathbf{l}(t) \\
\frac{d}{dt}\mathbf{v}(\mathbf{X}(t), t) &= (D_t \mathbf{v})(\mathbf{X}(t), t) \\
&= -\boldsymbol{\nabla} p(\mathbf{X}(t), t) + \nu \triangle \mathbf{v}(\mathbf{X}(t), t)
\end{aligned}$$

$$\begin{aligned}
d\mathbf{l}(t + \Delta t) &= d\mathbf{l}(t) + \Delta t \, (d\mathbf{l}(t) \cdot \boldsymbol{\nabla})\mathbf{v}(\mathbf{r}, t) \\
\therefore \;\; d\dot{\mathbf{l}}(t) &= (d\mathbf{l}(t) \cdot \boldsymbol{\nabla})\mathbf{v}(\mathbf{X}(t), t)
\end{aligned}$$

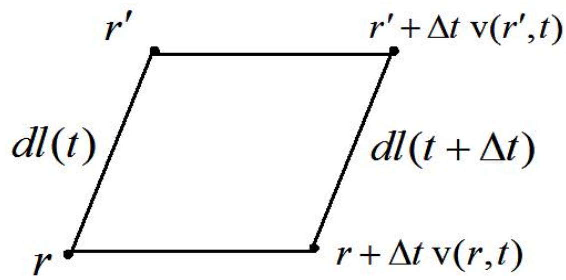

Figure 4. Proof of Kelvin theorem

Finally,

$$\begin{aligned}
\frac{d\Gamma_{C(t)}}{dt} &= \oint [-\boldsymbol{\nabla} p \cdot d\mathbf{l} + \mathbf{v} \cdot (d\mathbf{l} \cdot \boldsymbol{\nabla})\mathbf{v} + \nu \triangle \mathbf{v} \cdot d\mathbf{l}] \\
&= \oint_{C(t)} \boldsymbol{\nabla} \cdot (\frac{1}{2} v^2 - p) \cdot d\mathbf{l} + \nu \oint_{C(t)} \triangle \mathbf{v} \cdot d\mathbf{l} \\
&= \nu \oint_{C(t)} \triangle \mathbf{v} \cdot d\mathbf{l}
\end{aligned}$$

Taylor's Vortex-Stretching Picture

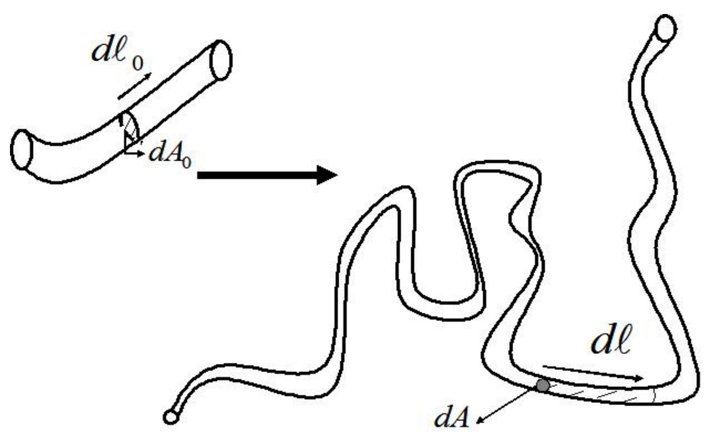

Figure 5. A vortex tube is being stretched.

$$\frac{\omega}{\omega_0} = \frac{\omega dA d\ell}{\omega_0 dA_0 d\ell_0}$$

$$\begin{array}{rl} \because & dAd\ell = dA_0 d\ell_0 \text{ from incompressibility} \\ \text{and since also} & \omega dA = \omega_0 dA_0 \text{ from Kelvin-Helmholtz} \\ \therefore & \dfrac{\omega}{\omega_0} = \dfrac{d\ell}{d\ell_0} \gg 1 \end{array}$$

Since the volume of the tube is also conserved,

$$\int \omega^2(t) d^3x \gg \int \omega_0^2 d^3x$$

See G.I. Taylor & A.E. Green, "Mechanism of the productivity of small eddies from large ones," Proc. Roy. Soc. Ser. A, **158**, 499 (1937).

Does this argument justify

$$\begin{aligned} \lim_{\nu \to 0} \varepsilon &= \lim_{\nu \to 0} \nu \int \omega^2 d^3x \\ &\neq 0? \end{aligned}$$

<u>Problem:</u>

$$\begin{aligned} \frac{d}{dt}\Gamma_c(t) &= \nu \oint_{C(t)} \triangle \mathbf{v} \cdot d\mathbf{l} \\ &\to 0 \text{ when } \nu \to 0??? \end{aligned}$$

In fact, circulations are *not* conserved in high Reynolds number turbulence, except in some average sense! Although Taylor's idea is doubtless part of the final answer, the details are still not understood.

It is possible to neglect the effects of viscosity on the large-scales, as we discuss next...

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