I The Continuum Navier-Stokes Equations

(A) Foundations of Macroscopic Hydrodynamics

See also Tennekes & Lumley, Section 2.2

Hamiltonian equations of a simple molecular fluid:

$$
\begin{aligned}
H &= \sum_{n=1}^{N} \frac{p_n^2}{2m} + \frac{1}{2}\sum_{n\neq m}\Phi(\mathbf{r}_{nm}), \ \ \mathbf{r}_{nm} = \mathbf{r}_n - \mathbf{r}_m \\
\dot{\mathbf{r}}_n &= \frac{\partial H}{\partial \mathbf{p}_n} = \frac{\mathbf{p}_n}{m}, \\
\dot{\mathbf{p}}_n &= -\frac{\partial H}{\partial \mathbf{r}_n} = \sum_{m\neq n} -\boldsymbol{\nabla}\Phi(\mathbf{r}_{nm}), \ \ -\boldsymbol{\nabla}\Phi(\mathbf{r}_{nm}) = \mathbf{F}_{nm}.
\end{aligned}
$$

Conservation laws:

$$
\begin{aligned}
\underline{mass}: \ M &= mN \\
\underline{momentumn}: \ \mathbf{P} &= \sum_{n=1}^{N}\mathbf{p}_n \\
\underline{energy}: \ E &= \sum_{n=1}^{N}\frac{p_n^2}{2m} + \frac{1}{2}\sum_{n\neq m}\Phi(\mathbf{r}_{nm})
\end{aligned}
$$

Local conservation laws

mass density:

$$
\hat{\rho}(\mathbf{x},t) = \sum_{n=1}^{N} m\delta^3(\mathbf{x} - \mathbf{r}_n(t))
$$

mass current:

$$
\hat{\mathbf{g}}(\mathbf{x},t) = \sum_{n=1}^{N} \mathbf{p}_n(t)\delta^3(\mathbf{x} - \mathbf{r}_n(t))
$$

$$
\Longrightarrow \partial_t\hat{\rho}(\mathbf{x},t) + \boldsymbol{\nabla}\cdot\hat{\mathbf{g}}(\mathbf{x},t) = 0
$$

momentum density:

$$\hat{\mathbf{g}}(\mathbf{x}, t) = \sum_{n=1}^{N} \mathbf{p}_n(t) \delta^3(\mathbf{x} - \mathbf{r}_n(t))$$

momentum current(stress tensor):

$$
\begin{aligned}
\hat{\mathbf{T}}(\mathbf{x}, t) &= \sum_{n=1}^{N} \frac{1}{m} \mathbf{p}_n(t) \mathbf{p}_n(t) \delta^3(\mathbf{x} - \mathbf{r}_n(t)) \\
&+ \frac{1}{2} \sum_{n \neq m} \mathbf{F}_{nm} \mathbf{r}_{nm}(t) \int_0^1 ds \; \delta^3(\mathbf{x} - \mathbf{r}_n(t) + s \mathbf{r}_{nm}(t))
\end{aligned}
$$

$$\implies \partial_t \hat{\mathbf{g}}(\mathbf{x}, t) + \boldsymbol{\nabla} \cdot \hat{\mathbf{T}}(\mathbf{x}, t) = 0$$

energy density

$$\hat{e}(\mathbf{x}, t) = \sum_{n=1}^{N} \left(\frac{p_n^2}{2m} + \frac{1}{2} \sum_{m \neq n} \Phi(\mathbf{r}_{nm}) \right) \delta^3(\mathbf{x} - \mathbf{r}_n(t))$$

energy current

$$
\begin{aligned}
\hat{\mathbf{s}}(\mathbf{x}, t) &= \sum_{n=1}^{N} \left(\frac{p_n^2}{2m} + \frac{1}{2} \sum_{m \neq n} \Phi(\mathbf{r}_{nm}) \right) \frac{\mathbf{p}_n}{m} \delta^3(\mathbf{x} - \mathbf{r}_n(t)) \\
&+ \frac{1}{4} \sum_{n \neq m} \frac{1}{m} [(\mathbf{p}_n + \mathbf{p}_m) \cdot \mathbf{F}_{nm}] \mathbf{r}_{nm} \int_0^1 ds \; \delta^3(\mathbf{x} - \mathbf{r}_n + s \mathbf{r}_{nm})
\end{aligned}
$$

$$\implies \partial_t \hat{e}(\mathbf{x}, t) + \boldsymbol{\nabla} \cdot \hat{\mathbf{s}}(\mathbf{x}, t) = 0$$

Hydrodynamic regime: A sufficiently short-range potential $\Phi \implies$ the typical distance λ_{mfp} travelled by a molecule between $O(1)$ interactions is $\underline{finite}$. Here, λ_{mfp} is the *mean-free path*. Let

$$G_\ell(\mathbf{r}) = \ell^{-3} G(\frac{\mathbf{r}}{\ell})$$

2

for G smooth, rapidly decreasing in $|\mathbf{r}|$, positive, with unit integral. With

$$\hat{\boldsymbol{\rho}}(\mathbf{x}, t) = (\hat{\rho}_0, \hat{\rho}_1, \hat{\rho}_2, \hat{\rho}_3, \hat{\rho}_4) = (\hat{\rho}, \hat{g}_1, \hat{g}_2, \hat{g}_3, \hat{e})$$

set

$$\overline{\boldsymbol{\rho}}_\ell(\mathbf{x}, t) = (G_\ell \star \hat{\boldsymbol{\rho}})(\mathbf{x}, t) = \int d\mathbf{r} \ \hat{\boldsymbol{\rho}}(\mathbf{x} + \mathbf{r}, t) G_\ell(\mathbf{r})$$

For example,

$$\overline{\rho}_{0\ell}(\mathbf{x}, t) = \overline{\rho}_\ell(\mathbf{x}, t) = \sum_{n=1}^{N} m G_\ell(\mathbf{x} - \mathbf{r}_n(t))$$

Thus, $\overline{\boldsymbol{\rho}}_\ell(\mathbf{x}, t)$ is the conserved density averaged over all the molecules in a region of radius $\sim \ell$ around the point $\mathbf{x}$. These are called $\underline{\text{``}coarse - grained\ densities\text{''}}$ at the length-scale ℓ.
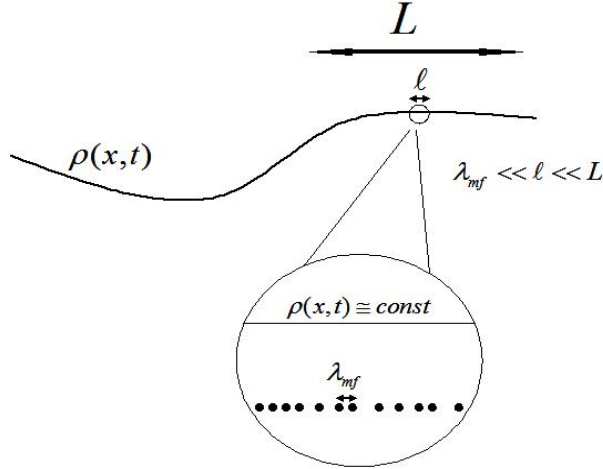
Figure 1.

Consider situations for which there exist smooth macroscopic densitities $\boldsymbol{\rho}(\mathbf{x}, t)$, with

$$
\begin{aligned}
L_\nabla &= max \ \frac{|\boldsymbol{\rho}|}{|\boldsymbol{\nabla}\boldsymbol{\rho}|} \gg \lambda_{mfp} \\
T_\nabla &= max \ \frac{|\boldsymbol{\rho}|}{|\boldsymbol{\rho}_t|} \gg \tau_{mfp} = \lambda_{mfp}/v_{th}
\end{aligned}
$$

with v_{th} the typical molecular speed (a thermal velocity) such that

$$\overline{\boldsymbol{\rho}}_\ell(\mathbf{x}, t) \cong \boldsymbol{\rho}(\mathbf{x}, t)$$

Gas	Collision times $[10^{-10}$ sec]	Mean free Paths $[10^{-8}$ m]
He	2.2	27.45
Ar	2.5	9.88
CO_2	1.6	6.15
H_2	1.0	17.44
N_2	2.0	9.29
O_2	2.2	9.93

Table 1: Ideal gases at temperature $T = 120°C$ and pressure $p = 1\ atm$.

for all $\lambda_{mfp} \ll \ell \ll L_\nabla$. This is called a situation with separation of scales. The small parameter $\varepsilon = Kn = \frac{\lambda_{mfp}}{L_\nabla} \ll 1$ is called the $\underline{Knudsen\ number}$. Since maximum macroscopic velocities are of the order of the sound speed which is $\cong v_{th}$, $\tau_{mfp}/T_\nabla \cong \varepsilon$.

Typically, on a timescale $O(\tau_{mfp})$, the particle distributions of positions and momentum in the region of radius ℓ achieve a $\underline{local\ equilibrium\ distribution}$

$$P^{(0)} \propto \frac{1}{Z} \exp\{-\beta(\mathbf{x}, t)[H_\ell - \mathbf{v}(\mathbf{x}, t) \cdot \mathbf{P}_\ell - \mu(\mathbf{x}, t)N_\ell]\}$$

with N_ℓ, $\mathbf{P}_\ell$, H_ℓ (i.e. $N_\ell \cong \frac{4}{3}\pi\ell^3\overline{\rho}_\ell$, etc) the particle number, total momentum and energy of the particles inside the region of radius ℓ at $\mathbf{x}$, t.

$$
\begin{aligned}
\beta(\mathbf{x}, t) &= \frac{1}{k_B T(\mathbf{x}, t)} \text{ is inverse temperature} \\
\mathbf{v}(\mathbf{x}, t) &\text{ is } \text{ macroscopic velocity} \\
\mu(\mathbf{x}, t) &\text{ is } \text{ chemical potential}
\end{aligned}
$$

The potentials β, μ, $\mathbf{v}$ uniquely determine the densities ρ, $\mathbf{g}$, e by equilibrium thermodynamics, and $vice\ versa$. Note in particular that, integrating over positions, the particle momenta have a very simple Maxwellian (i.i.d. Gaussian) distribution

$$P^{(0)}(\{\mathbf{p}_n | \text{for } n \text{ s.t. } \mathbf{x}_n(t) \in B_\ell(\mathbf{x})\}) \propto \exp\left(-\beta(\mathbf{x}, t)\sum_n \frac{p_n^2}{2m}\right).$$

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Space-ergodicity of the local-equilibrium statistics ties together the spatial coarse-graining and the ensemble average over the equilibrium distribution $P^{(0)}$:

$$\bar{f}_\ell(\mathbf{x},t) \longrightarrow f^{(0)}(\mathbf{x},t) = \langle f \rangle^{(0)}_{\beta(\mathbf{x},t),\mu(\mathbf{x},t),\mathbf{v}(\mathbf{x},t)}, \quad \ell \gg \xi, \tag{A}$$

where ξ is a correlation length of $P^{(0)}$ and $f_\omega(\mathbf{x},t)$ is any "local density", i.e. a function of $\mathbf{x}$, t and phase point

$$\omega = (\mathbf{r}_n, \mathbf{p}_n), n = 1, \ldots, N$$

such that $f_\omega(\mathbf{x},t)$ depends only upon the particles with location nearby $\mathbf{x}$ at the t. For example, a strictly local density would satisfy $f_\omega(\mathbf{x},t) = f_{\omega_{B(\mathbf{x},R)}}(\mathbf{x},t)$ where $\omega_{B(\mathbf{x},R)}$ is the set $\{(\mathbf{r}_n, \mathbf{p}_n), \mathbf{r}_n \in B(\mathbf{x},R): \ n = 1, \ldots, N\}$, so that R is the range of f. We also require that

$$f_{\sigma_\mathbf{a}\omega}(\mathbf{x}+\mathbf{a},t) = f_\omega(\mathbf{x},t)$$

where $\sigma_\mathbf{a}\omega = \{(\mathbf{r}_n + \mathbf{a}, \mathbf{p}_n), n = 1, \ldots, N\}$. Then (A) holds almost surely for every ω with respect to $P^{(0)}$. See:

H. Spohn, *Large-Scale Dynamics of Interacting Particles* (Springer, 1991)

A. Martin-Löf, *Statistical Mechanics and the Foundations of Thermodynamics* (Springer, 1979).

The mean currents (fluxes) $\mathbf{J}_\ell(\mathbf{x},t) \cong \mathbf{j}^{(0)}(\mathbf{x},t)$ which satisfy

$$\partial_t \rho_a(\mathbf{x},t) + \boldsymbol{\nabla} \cdot \mathbf{j}_a^{(0)}(\mathbf{x},t) = 0, \ a = 0, 1, 2, 3, 4.$$

are also given by equilibrium thermodynamics

$$\begin{aligned}
\mathbf{g}^{(0)}(\mathbf{x},t) &= \mathbf{j}_0^{(0)}(\mathbf{x},t) = \rho(\mathbf{x},t)\mathbf{v}(\mathbf{x},t)) \\
\hat{\mathbf{e}}_i \cdot \mathbf{T}^{(0)}(\mathbf{x},t) &= \mathbf{j}_i^{(0)}(\mathbf{x},t) = \rho(\mathbf{x},t)v_i(\mathbf{x},t)\mathbf{v}(\mathbf{x},t) + P(\mathbf{x},t)\hat{\mathbf{e}}_i \\
\mathbf{s}^{(0)}(\mathbf{x},t) &= \mathbf{j}_4^{(0)}(\mathbf{x},t) = (u(\mathbf{x},t) + P(\mathbf{x},t) + \frac{1}{2}\rho(\mathbf{x},t)v^2(\mathbf{x},t))\mathbf{v}(\mathbf{x},t) \\
&= (e(\mathbf{x},t) + P(\mathbf{x},t))\mathbf{v}(\mathbf{x},t)
\end{aligned}$$

where $P = P(\rho, u)$ is the thermodynamic pressure. Note that the total energy density is the sum of $u(\mathbf{x}, t)$, the internal energy per volume, and the kinetic energy density:

$$e(\mathbf{x}, t) = u(\mathbf{x}, t) + \frac{1}{2}\rho(\mathbf{x}, t)v^2(\mathbf{x}, t).$$

The equations

$$
\begin{aligned}
\partial_t \rho + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) &= 0, \\
\partial_t(\rho \mathbf{v}) + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}\mathbf{v} + P\mathbf{I}) &= 0, \\
\partial_t e + \boldsymbol{\nabla} \cdot [(e + P)\mathbf{v}] &= 0.
\end{aligned}
$$

are the compressible Euler equations. There are closed equations for the densities ρ, $\mathbf{g}$, e. Closure has been achieved because of the local equilibrium distribution of the molecules in the small regions of radius ℓ.

Navier-Stokes equation of a simple fluid

There is, however, an $O(\varepsilon)$ correction to local equilibruim

$$\mathbf{J}_\ell(\mathbf{x}, t) \cong \mathbf{j}^{(0)}(\mathbf{x}, t) + \mathbf{j}^{(1)}(\mathbf{x}, t), \ \ \mathbf{j}^{(1)}(\mathbf{x}, t) \sim O(\varepsilon)$$

where*

$$
\begin{aligned}
\mathbf{g}^{(1)} &\equiv 0 \ \text{(by momentum conservation!)} \\
\mathbf{T}^{(1)} &= -\eta \left[\boldsymbol{\nabla}\mathbf{v} + (\boldsymbol{\nabla}\mathbf{v})^\top - \frac{2}{3}(\boldsymbol{\nabla} \cdot \mathbf{v})\mathbf{I} \right] - \zeta(\boldsymbol{\nabla} \cdot \mathbf{v})\mathbf{I} \\
\mathbf{s}^{(1)} &= \mathbf{T}^{(1)} \cdot \mathbf{v} - \kappa\boldsymbol{\nabla}T.
\end{aligned}
$$

Here heat current $q^{(1)} = -\kappa\boldsymbol{\nabla}T$. The following *transport coefficients* appear

$$
\begin{aligned}
\eta &= \ \text{shear viscosity;} \\
\zeta &= \ \text{bulk viscosity;} \\
\kappa &= \ \text{heat conductivity or thermal conductivity.}
\end{aligned}
$$

with η, ζ, $\kappa \geq 0$.

$\star$ Note: In principle, there is another rotational viscosity η_{rot} and an additional stress term $\mathbf{T}^{(1)}_{rot} = -\eta_{rot}[\boldsymbol{\nabla}\mathbf{v} - (\boldsymbol{\nabla}\mathbf{v})^{\top} - \boldsymbol{\Omega}]$ where $\Omega_{ij} = -\frac{1}{2}\epsilon_{ijk}\omega_k$ and $\boldsymbol{\omega}$ is the local angular velocity of the fluid. However, this term can generally be neglected!

<u>Kinetic Theory derivation</u>

For a linear shear profile $u(y)$, consider transport of $x-$momentum along the direction y
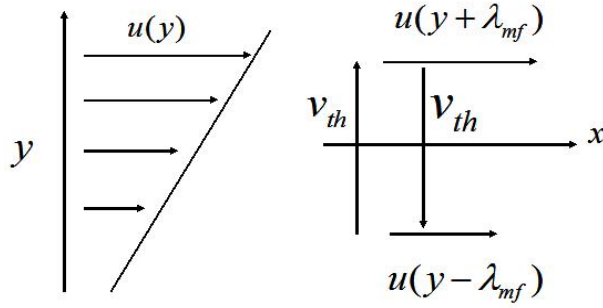

Figure 2.

$T^{(1)}_{xy}$ is the net flux of $x-$momentum in the $y-$direction

$$
\begin{aligned}
T^{(1)}_{xy} &= \rho u(y - \lambda_{mfp})v_{th} - \rho u(y + \lambda_{mfp})v_{th} \\
&\cong -2\rho\lambda_{mfp}v_{th}\frac{\partial u}{\partial y}(y) \\
&= -\eta\frac{\partial u}{\partial y}
\end{aligned}
$$

where $\eta \cong \rho\lambda_{mfp}v_{th}$.

Note that compared with $\mathbf{T}^{(0)} = \rho\mathbf{v}\mathbf{v} + P\mathbf{I}$

$$
\frac{|\mathbf{T}^{(1)}|}{|\mathbf{T}^{(0)}|} \cong \frac{\rho\lambda_{mfp}v_{th}\frac{u}{L}}{\rho u^2} = \frac{\lambda_{mfp}}{L}\frac{v_{th}}{u} = O(\varepsilon)
$$

The final equations

$$\partial_t \rho + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = 0,$$

$$\partial_t (\rho \mathbf{v}) + \boldsymbol{\nabla} \cdot \left(\rho \mathbf{v} \mathbf{v} + P\mathbf{I} + \mathbf{T}^{(1)} \right) = 0,$$

$$\partial_t e + \boldsymbol{\nabla} \cdot \left[(u + P + \frac{1}{2}\rho v^2)\mathbf{v} + \mathbf{T}^{(1)} \cdot \mathbf{v} + \mathbf{q}^{(1)} \right] = 0$$

with

$$
\begin{aligned}
\mathbf{T}^{(1)} &= -\eta \left[\boldsymbol{\nabla}\mathbf{v} + (\boldsymbol{\nabla}\mathbf{v})^T - \frac{2}{3}(\boldsymbol{\nabla} \cdot \mathbf{v})\mathbf{I} \right] - \zeta(\boldsymbol{\nabla} \cdot \mathbf{v})\mathbf{I} \\
\mathbf{q}^{(1)} &= -\kappa \boldsymbol{\nabla} T
\end{aligned}
$$

are the compressible Navier-Stokes equations, withh $\eta(\rho, T)$, $\zeta(\rho, T)$, $\kappa(\rho, T)$ functions of mass density ρ and temperature T (given by so-called Green-Kubo formulas).

Remarks:

(i) Like the Euler equations, the Navier-Stokes equations are closed hydrodynamic equations in terms of $(\rho, \mathbf{g}, e)$ [or $(\rho, \mathbf{v}, e)$]. Closure is achieved because not only is the statistics of the molecules given by local equilibrium to $O(\varepsilon^0)$, but also the $O(\varepsilon^1)$ corrections are determined by the gradients of $(\rho, \mathbf{v}, e)$. To get closure, separation of scales and small $\varepsilon = \frac{\lambda_{mfp}}{L_{\boldsymbol{\nabla}}}$ is crucial.

(ii) There are also small random terms to $O(\varepsilon^1)$ which we have neglected

$$
\begin{aligned}
\mathbf{T}^{(1)} &= -\eta \left[\boldsymbol{\nabla}\mathbf{v} + (\boldsymbol{\nabla}\mathbf{v})^\top - \frac{2}{3}(\boldsymbol{\nabla} \cdot \mathbf{v})\mathbf{I} \right] - \zeta(\boldsymbol{\nabla} \cdot \mathbf{v})\mathbf{I} + \tilde{\mathbf{T}}^{(1)} \\
\mathbf{q}^{(1)} &= -\kappa \boldsymbol{\nabla} T + \tilde{\mathbf{q}}^{(1)}
\end{aligned}
$$

where the stochastic fluxes $\tilde{\mathbf{T}}^{(1)}$, $\tilde{\mathbf{q}}^{(1)}$ are called molecular noise. They represent the chaotic effects of the unknown molecular positions and momentum. The covariances of the random terms are given by fluctuation-dissipation relations in terms of transport coefficients η, ζ, κ:

$$\tilde{T}_{ij}^{(1)} = \sqrt{2k_B \eta T}\, \eta_{ij} + \sqrt{2k_B \zeta T}\, \eta'_{ij}, \quad \tilde{q}_i^{(1)} = \sqrt{2k_B \kappa T^2}\, \eta_i$$

where η_{ij}, η'_{ij}, η_i are space-time Gaussian white-noises with zero mean and covariances

$$\langle \eta_{ij}(\mathbf{x}, t)\eta_{kl}(\mathbf{x}', t') \rangle = \left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{1}{3}\delta_{ij}\delta_{kl} \right) \delta^3(\mathbf{x} - \mathbf{x}')\delta(t - t')$$

$$\langle \eta'_{ij}(\mathbf{x},t)\eta'_{kl}(\mathbf{x}',t')\rangle = \delta_{ij}\delta_{kl}\delta^3(\mathbf{x}-\mathbf{x}')\delta(t-t'), \quad \langle \eta_i(\mathbf{x},t)\eta_j(\mathbf{x}',t')\rangle = \delta_{ij}\delta^3(\mathbf{x}-\mathbf{x}')\delta(t-t').$$

Thus, the statistics of these terms are also given, once ρ, $\mathbf{v}$, e are known. Such equations of fluctuating hydrodynamics were first proposed by

> L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, Course of Theoretical Physics,
> Vol. 6 (Pergamon Press, 1959)

A fundamental reference is

> V. G. Morozov, "On the Langevin formalism for nonlinear and nonequilibrium hy-
> drodynamic fluctuations," Physica A **126** 443-460 (1984)

which first derived the correct fluctuation equations for a compressible Navier-Stokes fluid. It is important to emphasize that, despite appearances, these are not stochastic PDE's! Instead, all fields and delta functions are cut off at some high wavenumber Λ, where $L_\nabla^{-1} \ll \Lambda \ll \lambda_{mfp}^{-1}$.

Even somewhat earlier an essentially equivalent formulation by an Onsager-Machlup "effec-tive action" had been proposed by Robert Graham, see:

> R. Graham, "Onset of cooperative behavior in nonequilibrium steady states", in
> *Order and Fluctuations in Equilibrium and Nonequilibrium Statistical Mechanics*,
> G. Nicolis, G. Dewel, and J. W. Turner, eds. (Wiley, New York, 1981).
> G. L. Eyink, "Dissipation and large thermodynamic fluctuations," J. Stat. Phys.
> **61** 533-572 (1990)

We shall see that thermal fluctuations are non-negligible at very small scales in turbulent flows! In fact, this was first noted in pioneering papers of Robert Betchov:

> R. Betchov, "On the fine structure of turbulent flows," Journal of Fluid Mechanics
> **3**, 205-216 (1957)
> R. Betchov, "Thermal Agitation and Turbulence," in: *Rarefied Gas Dynamics*, Pro-
> ceedings of the 2nd International Symposium, ed. L. Talbot. (Academic Press, New
> York, 1961), pp.307-321

We shall discuss this subject later after developing a deeper understanding of turbulence.

<u>Energy balance</u>

From

$$\partial_t(\rho\mathbf{v}) + \boldsymbol{\nabla}\cdot[\rho\mathbf{v}\mathbf{v} + P\mathbf{I} + \mathbf{T}^{(1)}] = 0$$

one easily derives

$$\partial_t(\tfrac{1}{2}\rho v^2) + \boldsymbol{\nabla}\cdot\left[(\tfrac{1}{2}\rho v^2 + P)\mathbf{v} + \mathbf{T}^{(1)}\cdot\mathbf{v}\right] \;=\; -Q + P(\boldsymbol{\nabla}\cdot\mathbf{v})$$

with $Q = -\mathbf{T}^{(1)} : \boldsymbol{\nabla}\mathbf{v} = 2\eta|\mathbf{S}|^2 + \zeta(\boldsymbol{\nabla}\cdot\mathbf{v})^2 \geq 0$ where $\mathbf{S} = \mathring{\boldsymbol{\nabla}}\mathbf{v}^s = \tfrac{1}{2}[\boldsymbol{\nabla}\mathbf{v} + (\boldsymbol{\nabla}\mathbf{v})^\top] - \tfrac{1}{3}(\boldsymbol{\nabla}\cdot\mathbf{v})\mathbf{I}$.

Thus, <u>kinetic energy is not conserved!</u>

However, <u>total energy</u> *is* conserved

$$\partial_t e + \boldsymbol{\nabla}\cdot\left[(u + P + \tfrac{1}{2}\rho v^2)\mathbf{v} + \mathbf{T}^{(1)}\cdot\mathbf{v} + \mathbf{q}^{(1)}\right] = 0$$

$\Longrightarrow$ internal energy $u = e - \tfrac{1}{2}\rho v^2$ satisfies

$$\partial_t u + \boldsymbol{\nabla}\cdot(u\mathbf{v} + \mathbf{q}^{(1)}) = -\mathbf{T}^{(1)} : \boldsymbol{\nabla}\mathbf{v} - P(\boldsymbol{\nabla}\cdot\mathbf{v})$$

Hence, the viscous energy dissipation appears as a sink for kinetic energy and as a source for internal energy. As the fluid slows down, it also heats up!

Another very important property of the Navier-Stokes system has to do with <u>thermodynamic entropy</u> $s(u,\rho)$ which satisfies the equation

$$\partial_t s + \boldsymbol{\nabla}\left[s\mathbf{v} + \mathbf{q}^{(1)}/T\right] = \frac{Q}{T} + \frac{\kappa|\boldsymbol{\nabla}T|^2}{T^2} \geq 0.$$

This is a local form of the <u>second law of thermodynamics.</u>

<u>Incompressible Limit</u>

Another important simplification appears if the typical fluid velocities U are much less than the sound speed $c \cong v_{th}$. This corresponds to a limit of small <u>Mach number</u>

$$Ma \equiv \frac{U}{c}.$$

Sound waves are compressible waves with periodic variation of the mass density ρ. When $c \gg U$, then difference in density are adjusted very quickly compared with fluid motions. Thus,

$$\rho \cong constant$$

In that case

$$\partial_t \rho + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = 0 \Longrightarrow \rho \boldsymbol{\nabla} \cdot \mathbf{v} = 0 \Longrightarrow \boldsymbol{\nabla} \cdot \mathbf{v} = 0$$

The viscous stress becomes

$$
\begin{aligned}
\mathbf{T}^{(1)} &= -\eta[\boldsymbol{\nabla}\mathbf{v} + (\boldsymbol{\nabla}\mathbf{v})^\top - \frac{2}{3}(\boldsymbol{\nabla}\cdot\mathbf{v})\mathbf{I}] - \zeta(\boldsymbol{\nabla}\cdot\mathbf{v})\mathbf{I} \\
&= -2\eta\mathbf{S}
\end{aligned}
$$

with $\mathbf{S} = \frac{1}{2}[\boldsymbol{\nabla}\mathbf{v} + (\boldsymbol{\nabla}\mathbf{v})^\top]$ the strain(rate) field since $\boldsymbol{\nabla}\cdot\mathbf{v} = 0$ and*

$$\boldsymbol{\nabla}\cdot\mathbf{T}^{(1)} = -\eta[\triangle\mathbf{v} + \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{v})] = -\eta\triangle\mathbf{v}$$

$\star$ Weak variations of η with temperature have been neglected here!

Thus, the momentum equation becomes

$$
\begin{aligned}
\partial_t(\rho\mathbf{v}) + \boldsymbol{\nabla}\cdot[\rho\mathbf{v}\mathbf{v} + P\mathbf{I} + \mathbf{T}^{(1)}] &= 0 \\
\Longrightarrow \rho\partial_t\mathbf{v} + \rho[(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} + (\boldsymbol{\nabla}\cdot\mathbf{v})\mathbf{v}] + \boldsymbol{\nabla}P - \eta\triangle\mathbf{v} &= 0 \\
\Longrightarrow \partial_t\mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = -\boldsymbol{\nabla}p + \nu\triangle\mathbf{v}, \ \boldsymbol{\nabla}\cdot\mathbf{v} &= 0
\end{aligned}
$$

the <u>incompressible Navier-Stokes equation</u> with $p = \frac{P}{\rho}$ the kinetic pressure; $\nu = \frac{\eta}{\rho}$ the kinetic viscosity. Note that the (kinematic) pressure is determined from the <u>Poisson equation</u>

$$-\nabla p = \boldsymbol{\nabla}\cdot[(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}] = (\partial_i v_j)(\partial_j v_i)$$

using $\boldsymbol{\nabla}\cdot\mathbf{v} = 0$

The energy equation also satisfies

$$
\begin{aligned}
\partial_t u + \boldsymbol{\nabla}\cdot(u\mathbf{v} + \mathbf{q}^{(1)}) &= -\mathbf{T}^{(1)}:\boldsymbol{\nabla}\mathbf{v} - P(\boldsymbol{\nabla}\cdot\mathbf{v}) \\
\Longrightarrow (\partial_t + \mathbf{v}\cdot\boldsymbol{\nabla})u - \kappa\triangle T &= 2\eta|\mathbf{S}|^2
\end{aligned}
$$

This equation for internal energy can be written as one for <u>temperature</u>, using the definition⋆

$$c_p = \frac{1}{\rho}(\frac{\partial u}{\partial T})_p = \text{ specific heat at constant pressure (per unit mass)}$$

$$\implies \quad \rho c_p(\partial_t + \mathbf{v} \cdot \boldsymbol{\nabla})T = \kappa \triangle T + 2\eta|\mathbf{S}|^2$$
$$\implies \quad (\partial_t + \mathbf{v} \cdot \boldsymbol{\nabla})T = \lambda_T \triangle T + \frac{\varepsilon}{c_p}$$

with

$$\lambda_T = \frac{\kappa}{\rho c_p} = \text{ thermal diffusivity}$$
$$\varepsilon = 2\nu|\mathbf{S}|^2 = \text{ energy dissipation per unit mass.}$$

⋆ In fact, one gets directly the stated result with instead the specific heat at constant volume $c_v = \frac{1}{\rho}(\frac{\partial u}{\partial T})_\rho$, but $c_v = c_p$ for an incompressible fluid. The result as stated is more generally valid for a so-called Boussinesq fluid, taking into account some compressibility effects.

<u>NOTES & REFERENCES</u>

<u>Nonequilibrium Thermodynamics</u>

deGroot, S. R. & Mazur P., Nonequilibrium Thermodynamics, *Dover Publishing*, New York, 1984.

Kreuzer, H.J., Nonequilibrium Thermodynamics and its Statistical Foundations, *Clarendon Press*, Oxford, 1981.

<u>Nonequilibrium Statistical Mechanics</u>

Zubarev, D. N., Nonequilibrium Statistical Thermodynamics, *Plenum*, New York, 1974.

McLennan, J. A., Nonequilibrium Statistical Mechanics, *Prentice Hall*, Englewood Cliffs, New York, 1989.

In particular, for the microscopic derivation of the Landau-Lifschitz equations, see:

Zubarev, D. N. and Morozov, V. G., "Statistical mechanics of nonlinear hydrodynamic fluctuations," Physica A **120** 411-467 (1983)

Español, P. "Stochastic differential equations for non-linear hydrodynamics," Physica A **248** 77-96 (1998)

Derivation of Incompressible Navier-Stokes

Klainerman S. & Majda, A.J.. Singular limits of quasilinear hyberbolic systems with large parameters and incompressible fluids, *Comm. Pure. App. Math*, **34**, 481-524, 1981

gives a rigorous derivation of incompressible Navier-Stokes from compressible, in the limit of small Mach number. See also:

Majda, A. J. Compressible Fluid Flow & Systems of Conservation Laws in Several Space Variables, *Springer*, New York, 1984

B. J. Bayly, C. D. Levermore, and T. Passot, Density variations in weakly compressible flows, Phys. Fluids A **4** 945–954, 1992

The following paper

Quastel, J. & Yan, H.T., Lattice gases, large deviations and the incompressible Navier-Stokes equations, *Ann. Math.*, **148**, 51-108, 1998.

gives a rigorous derivation of the incompressible Navier-Stokes equation from a microscropic particle model (a lattice gas) in the limit of small Knudsen and Mach numbers, for any fixed but arbitrarlly large Reynolds number. An important point of their derivation is that if applies even if the Navier-Stokes equation develops a singularity! In that case, a distributional solution in the sense of Leray describes the fluid motion. Furthermore, the authors derive not only the deterministic fluid equation with probability approaching one (law of large numbers) but they

also rigorously derive the nonlinear Onsager-Machlup action for the small thermal fluctuations (large deviations theorem).

A careful derivation of the low Mach-number fluctuating hydrodynamic equations for an incompressible fluid, starting from the compressible Landau-Lifschitz equations, is given by

A. Donev et al., "Low Mach number fluctuating hydrodynamics of diffusively mixing fluids," Commun. in Appl. Math. Comp. Sci. **9** 47-105 (2014).

<u>Finite Mach number Effects</u>

We focus mainly on low Mach number flows (a wide class). One important finite-Mach effect is sound generation by turbulence. See:

Lighthill, M. J., On sound generated aerodynamically: I. General Theory, *Proc. Roy. Soc. land. A*, **211**, 564-578,1952.

Miksis, M. J. & Ting, L., On vortical flow and sound generations, *SIAM J. App. Math.*, **50**, 521-536, 1990.

<u>Boussinesq Equations</u>

Normand, C., Pomeau, Y. & Velarde, M. G. Convective instability: a physicist's approach. *Rev. Mod. Phys.*, **49**, 581-627, 1977