(D) Conservation of Circulations in Turbulent Flow

We have emphasized the importance of developing a better understanding of the dynamical & statistical origin of the positivity of vortex-stretching rate

$$\langle \bar{\boldsymbol{\omega}}_\ell^\top \overline{\mathbf{S}}_\ell \bar{\boldsymbol{\omega}}_\ell \rangle \ > \ 0.$$

For the fine-grained vorticity ($\ell \to 0$), this is important in Taylor's proposed mechanism for production of energy dissipation $\langle \nu |\boldsymbol{\omega}|^2 \rangle$. For ℓ in the inertial-range, we have seen that the flux of energy to small-scales is proportional to the stretching rate:

$$\langle \Pi_\ell \rangle \propto \ell^2 \langle \bar{\boldsymbol{\omega}}_\ell^\top \overline{\mathbf{S}}_\ell \bar{\boldsymbol{\omega}}_\ell \rangle,$$

to a first approximation, local in space and in scale. A deeper understanding of the vortex-stretching process might help us to better predict and control turbulent flow.

As we reviewed at the beginning of this chapter, most arguments for positivity of vortex-stretching use — in some form or another — the conservation of circulations. Here we give, just as one instance, the following explanation of T&L:

> "Vorticity amplification is a result of the kinematics of turbulence. As an example, take a situation in which the principlal axes of the instantaneous strain rate are aligned with the coordinate system, so that S_{ij} has only diagonal components (S_{11}, S_{22},and S_{33}). Let us assume for simplicity that $S_{22} = S_{33}$, so that, by virtue of continuity, $S_{11} = -2S_{22}$. The term $\omega_i \omega_j S_{ij}$ becomes, if we also assume that $\omega_2^2 = \omega_3^2$,
>
> $$\omega_1^2 S_{11} + \omega_{22}^2 S_{22} + \omega_3^2 S_{33} = S_{11}(\omega_1^2 - \omega_2^2)$$
>
> If $S_{11} > 0$, ω_1^2 is amplified (see Figure 3.5), but ω_2^2 and ω_3^2 are attenuated because S_{22} and S_{33} are negative. Thus, $\omega_1^2 - \omega_2^2$ tends to become positive if S_{11} is positive. Again, if $S_{11} < 0$, ω_1^2 decreases, but ω_2^2 and ω_3^2 increase, so that $\omega_1^2 - \omega_2^2 < 0$, making the stretching term positive again."

— H. Tennekes & J. L. Lumley, A First Course in Turbulence, (MIT press, Cambridge, MA, 1972), Section 3.3. p.92

This argument assumes that, as the vortex lengthens along a particular direction — the 1-direction, say — the corresponding component ω_1 of vorticity increases. This result depends, however, on the conservation of circulations. T&L discuss this explicitly in their explanation of their Fig. 3.5, which illustrates vortex-stretching in a wind-tunnel contraction:

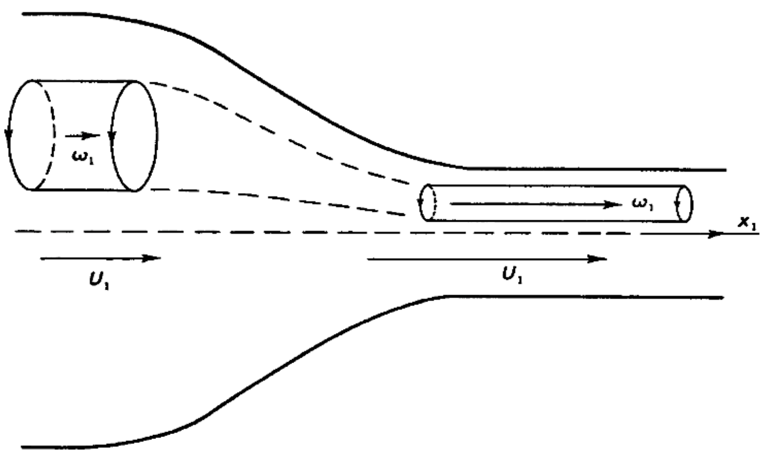


Figure 3.5. Vortex stretching in a wind-tunnel contraction. As the flow speeds up from left to right, the vorticity component ω_1 is amplified because angular momentum has to be conserved.

To explain this figure, T&L wrote:

"The change of vorticity by vortex stretching is a consequence of the conservation of angular momentum. The angular momentum of a material volume element would remain constant if viscous effects were absent; if the fluid element is stretched so that its cross-sectional area and moment of inertial become smaller, the component of the angular velocity in the direction of the stretching must increase in order to conserve angular momentum."

- H. Tennekes & J. L. Lumley, A First Course in Turbulence, (MIT press, Cambridge, MA, 1972), Section 3.3. pp. 83-84

As usual, T&L use the term "conservation of angular momentum" as a more elementary substitute for the more proper term "conservation of circulations." The argument is essentially that of G. I. Taylor (1937, 1938) that we discussed earlier. If we let ω_1, ω_1' represent the vorticity magnitude in the 1-direction before and after stretching, respectively, and likewise A_1, A_1' the cross-sectional area of the vortex tube before and after, conservation of circulations implies that

$$\omega_1 A_1 = \omega_1' A_1'.$$

If we let ℓ_1, ℓ_1' represent the length of the tube before and after stretching, then incompressibility implies

$$\ell_1 A_1 = \ell_1' A_1'.$$

It therefore follows that $\omega_1/\ell_1 = \omega_1'/\ell_1'$ or, equivalently,

$$\omega_1'/\omega_1 = \ell_1'/\ell_1.$$

Thus, the vorticity magnitude grows in direct propotion to line length.

Large-Scale Circulation Balance

The arguments of Taylor, Tennekes & Lumley, etc. assume that circulations will be conserved "if viscous effects were absent." The negligibility of the viscous terms may indeed be expected in the large scales, with ℓ fixed as $\nu \to 0$. Thus, let us consider in detail the balance of circulations in the scales $> \ell$.

We have seen that the momentum balance in the large scales takes the form

$$\begin{aligned} \overline{D}_{\ell t}\bar{\mathbf{u}}_\ell &= -\boldsymbol{\nabla}\bar{p}_\ell + \mathbf{f}_\ell^s - \nu\boldsymbol{\nabla}\times\bar{\boldsymbol{\omega}}_\ell + \bar{\mathbf{f}}_\ell^B \\ &= -\boldsymbol{\nabla}p_\ell^* + \mathbf{f}_\ell^v - \nu\boldsymbol{\nabla}\times\bar{\boldsymbol{\omega}}_\ell + \bar{\mathbf{f}}_\ell^B \end{aligned} \tag{32}$$

with $p_\ell^* = \bar{p}_\ell + k_\ell$. Let us now consider the Lagrangian circulation

$$\overline{K}_\ell(t) \equiv \oint_{\overline{C}_\ell(t)} \bar{\mathbf{u}}_\ell(t) \cdot d\mathbf{x} \tag{33}$$

where $\overline{C}_\ell(t)$ is the loop C advected by the large-scale velocity $\bar{\mathbf{u}}_\ell$ to time t. One can see from a couple of points of view that this is the interesting quantity to consider. First, mathematically, we note that

$$\frac{d}{dt}\overline{K}_\ell(t) \equiv \oint_{\overline{C}_\ell(t)} [\mathbf{f}_\ell^{s,v} - \nu \boldsymbol{\nabla} \times \bar{\boldsymbol{\omega}}_\ell + \bar{\mathbf{f}}_\ell^B] \cdot d\mathbf{x} \tag{34}$$

Secondly, from the physical point of view, $\overline{K}_\ell(t)$ is the quantity that an experimentalist would consider who wanted to test the validity of Kelvin's Theorem by means of fluid measurements at space resolution ℓ. Measuring $\bar{\mathbf{u}}_\ell(\mathbf{x}, t)$ [e.g. by holographic PIV or other techniques], he would then be able to construct $\overline{C}_\ell(t)$ by solving

$$\tfrac{d}{dt}\overline{C}_\ell(\theta_i, t) = \bar{\mathbf{u}}_\ell(\overline{C}_\ell(\theta_i, t), t)$$

for a set of fluid markers $\theta_i \in [0, 2\pi], i = 0, \cdots, N-1$. Finally, he could take

$$\overline{K}_\ell(t) \cong \sum_{i=0}^{N-1} \bar{\mathbf{u}}_\ell(\overline{C}_\ell(\theta_i, t), t) \cdot [\overline{C}_\ell(\theta_{i+1}, t) - \overline{C}_\ell(\theta_i, t)] \tag{35}$$

or some other discrete approximation to the integral, converging in the limit $N \to \infty$. By then taking ν as small as possible and measurements at finer resolution $\ell \to 0$, the experimentalist could attempt to verify the validity of conservation of circulations.

Now let us consider the order of magnitude of various terms in the circulation balance. First,

$$\mathbf{f}_\ell^{s,v} = O(\tfrac{\delta u^2(\ell)}{\ell})$$

as we have seen before, so that

$$\oint_{\overline{C}_\ell(t)} \mathbf{f}_\ell^{s,v}(t) \cdot d\mathbf{x} = O(\frac{\delta u^2(\ell)}{\ell} \cdot L(\overline{C}_\ell(t))) \tag{36}$$

where $L(C)$ is the length of the rectifiable curve C. Note that $\overline{C}_\ell(t)$ is rectifiable when the starting loop C is so, since $\bar{\mathbf{u}}_\ell$ is smooth and generates a flow of volume-preserving diffeomorphisms, which carries a rectifiable loop to another rectifiable loop.

Now consider the viscous term

$$-\nu \boldsymbol{\nabla} \times \bar{\boldsymbol{\omega}}_\ell = \nu \triangle \bar{\mathbf{u}}_\ell = O(\tfrac{\nu \delta u(\ell)}{\ell^2}) = O(\tfrac{\delta u^2(\ell)}{\ell} Re_\ell^{-1})$$

with $Re_\ell \equiv \ell \delta u(\ell)/\nu$. Thus,

$$-\nu \oint_{\overline{C}_\ell(t)} (\boldsymbol{\nabla} \times \bar{\boldsymbol{\omega}}_\ell) \cdot d\mathbf{x} = O(\frac{\delta u^2(\ell)}{\ell} L(\overline{C}_\ell(t)) \cdot Re_\ell^{-1}) \tag{37}$$

which is much smaller than the turbulent force term for $Re_\ell \gg 1$. Thus, we have some theoretical support to the idea that the viscous term is negligible. In particular, it vanishes in the limit as $\nu \to 0$ with ℓ fixed.

Now consider the contribution from the body force $\mathbf{f}^B$. To get the best estimate, it is helpful to transform this term using Stokes Theorem:

$$\oint_{\overline{C}_\ell(t)} \bar{\mathbf{f}}_\ell^B(t) \cdot d\mathbf{x} = \oint_{\text{minimal surface spanning } \overline{C}_\ell(t)} (\boldsymbol{\nabla} \times \bar{\mathbf{f}}_\ell^B) \cdot d\mathbf{A} \tag{38}$$

We have used the freedom in choosing the surface which spans the loop $\overline{C}_\ell(t)$ to select the one of minimal area. It follows from the work of

> J. Douglas, "Solution of the problem of Plateau," Trans. Ann. Math. Soc. **33** 263-321(1931)

that such a minimal spanning surface S exists for any closed simple curve C not necessarily even rectifiable. [For this work, Douglas won the first Fields Medal in 1936!] Thus,

$$\oint_{\overline{C}_\ell(t)} \bar{\mathbf{f}}_\ell^B(t) \cdot d\mathbf{x} = O(||\boldsymbol{\nabla} \times \bar{\mathbf{f}}_\ell^B||_\infty A(\overline{S}_\ell^{\min}(t))) \tag{39}$$

Note that $\boldsymbol{\nabla} \times \bar{\mathbf{f}}_\ell^B = \overline{(\boldsymbol{\nabla} \times \mathbf{f}^B)}_\ell$ so that $||\boldsymbol{\nabla} \times \bar{\mathbf{f}}_\ell^B||_\infty \leq ||\boldsymbol{\nabla} \times \mathbf{f}^B||_\infty$. Let us assume further that

$$\mathbf{f}^B(\mathbf{x}, t) = \int_{|\mathbf{k}| < \frac{2\pi}{L}} \hat{\mathbf{f}}^B(\mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{x}} \tag{40}$$

so that

$$||\boldsymbol{\nabla} \times \mathbf{f}^B||_{L^\infty} \leq \frac{2\pi}{L} ||\hat{\mathbf{f}}^B||_{L^1} = O(\frac{1}{L}) \tag{41}$$

for some large L of the order of the integral length. Thus, we see that $||\boldsymbol{\nabla}\times\bar{\mathbf{f}}_\ell^B||_\infty$ is bounded, independent of ℓ, and quite small.

On the contrary, the term

$$\overline{\Gamma}_\ell(C,t) \equiv -\oint_{\overline{C}_\ell(t)} \mathbf{f}_\ell^{v,B}(t)\cdot d\mathbf{x} = O(\frac{\delta u^2(\ell)}{\ell}L(\overline{C}_\ell(t))) \tag{42}$$

may diverge in the limit as $\ell\to 0$! This term represents the torque around the loop $\overline{C}_\ell(t)$ imposed by the subscale force $\mathbf{f}_\ell^s$ [or $\mathbf{f}_\ell^v$]. For example, in K41 theory,

$$\delta u(\ell)\sim(\varepsilon\ell)^{1/3}\Longrightarrow \tfrac{\delta u^2(\ell)}{\ell}\sim(\varepsilon)^{2/3}\ell^{-1/3}\to\infty, \text{ as } \ell\to 0$$

More generally, if $\mathbf{u}$ has Hölder exponent h

$$\delta u(\ell)\sim\ell^h\Longrightarrow\tfrac{\delta u^2(\ell)}{\ell}\sim(const.)\ell^{2h-1}$$

which only needs to vanish if $h>\frac{1}{2}$. This seems to have been first observed by

G. L. Eyink, "Turbulent cascade of circulations," C. R. Physique, **7** 449-455(2006)

Since there seem to be many points in a turbulent flow with $h<1/2$ (in particular, the most-probable value $h_*\cong 1/3$),we see that $\mathbf{f}_\ell^s$ diverges as $\ell\to 0$ for a large number of points!

Furthermore, it is believed that a loop $C(t)$ advected by such a rough velocity field becomes fractal with (Hausdorff) dimension $D>1$. For example, see

F. C. G. A. Nicolleau & A. Elmaihy, "Study of the development of three-dimensional sets of fluid particles and iso-concentration fields using kinematic simulations," J. Fluid Mech. **517** 229-249 (2004)

and many references therein. Note that

$$N_\ell(C(t))\cong\tfrac{L(\overline{C}_\ell(t))}{\ell}$$

where $N_\ell(C(t))$ is the number of balls of radius ℓ required to cover $C(t)$. thus,

$$N_\ell(C(t))\sim(\tfrac{L_0}{\ell})^D, \text{ as } \ell\to 0$$

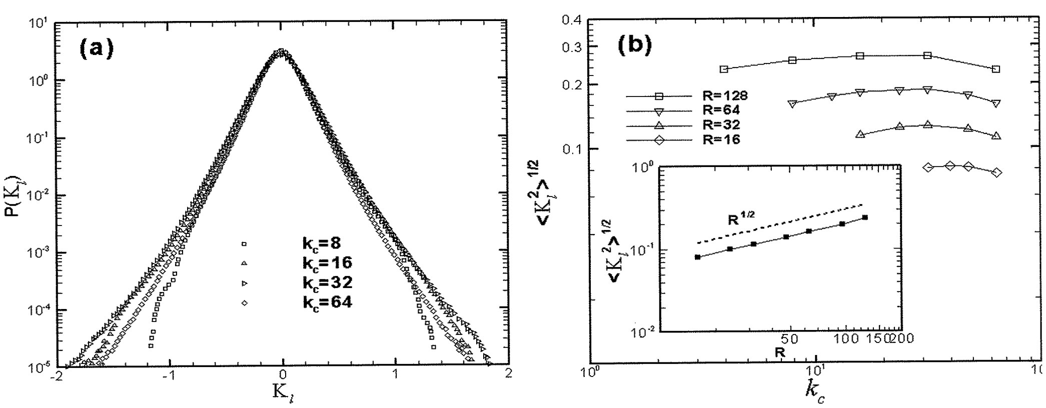
and

$$L(\overline{C}_\ell(t)) \sim \ell(\tfrac{L_0}{\ell})^D \sim \ell^{1-D}, \text{ as } \ell \to 0$$

This also diverges as $\ell \to 0$ since a fractal curve with $D > 1$ is non-rectifiable and has infinite length. Note, however, that $\overline{S}_\ell^{\min}(t)$ has finite area, so that $A(\overline{S}_\ell^{\min}(t))$ is constant for $\ell \ll L_0$. We thus see that the dominant term in the circulation balance is

$$\overline{\Gamma}_\ell(C,t) = -\oint_{\overline{C}_\ell(t)} \mathbf{f}_\ell^{v,B}(t) \cdot d\mathbf{x} = O(\delta u^2(\ell)(\frac{L_0}{\ell})^D) \tag{43}$$

as $\ell \to 0$, which leads to the distinct possibility that $\overline{\Gamma}_\ell(C,t)$ diverges in the limit. Of course, the RHS above is just a big-O bound or upper bound. There is the possibility of large cancellations in the integral over $\overline{C}_\ell(t)$, which could prevent divergence or even — in principle — allow $\overline{\Gamma}_\ell(C,t)$ to vanish in the limit $\ell \to 0$. To address this issue, we consider numerical results from S. Chen et al., "Is the Kelvin Theorem Valid for High Reynolds Number Turbulence?" PRL 97: 144505 (2006):



(a) PDF of the circulation flux for loops with radius $R = 64$ and for cutoff wave numbers $k_c = \pi/\ell$ with $\ell < R$.

(b) The rms value of the circulation flux as a function of k_c for various loop sizes R. The inset plots the plateau rms value versus R.

Thus, we see that

$$\overline{\Gamma}_\ell(C,t) \nrightarrow 0 \text{ as } \ell \to 0!$$

The effects of the subscale force are persistent as $\ell \to 0$. There is no conservation of circulations for fixed ℓ as $\nu \to 0$, although the effects of viscosity indeed become negligible in that limit.

Even as $\ell \to 0$ after having taken $\nu \to 0$ first, the effects of the subscale force term does not disappear! This casts doubt on the arguments of Taylor, Tennekes & Lumley, and others that appeal to conservation of circulations in explaining turbulent dynamics.

Similar results hold for the fine-grained circulation balance:

$$\frac{d}{dt}\oint_{C(t)} \mathbf{u}(t)\cdot d\mathbf{x} = \oint_{C(t)} [-\nu\boldsymbol{\nabla}\times\omega + \mathbf{f}^B]\cdot d\mathbf{x} \tag{44}$$

There is no subscale force, but now the viscous force $\mathbf{f}_\nu^{\text{vis}} = -\nu\boldsymbol{\nabla}\times\omega$ is large! This may be estimated as

$$f_\nu^{\text{vis}} = O(\nu\tfrac{\delta u(\eta_h)}{\eta_h^2}) = O(\tfrac{\nu u_0}{L^2}(\tfrac{\eta_h}{L})^{h-2}) = O(\tfrac{\nu u_0}{L}Re^{\frac{2-h}{1+h}})$$

since $\eta_h/L \sim (Re)^{-1/(1+h)}$ in the multifractal phenomenology. Then

$$f_\nu^{\text{vis}} = O(\tfrac{u_0^2}{L}Re^{\frac{1-2h}{1+h}})$$

at a point with Hölder exponent h. This diverges as $Re \to \infty$ unless $h > \frac{1}{2}$!!

We see thus also that the fine-grained circulations will not be conserved in a turbulent flow, even in the limit as $\nu \to 0$ (at least not in the conventional sense.) This was appreciated, to some extent, by G. I. Taylor. For example, he wrote

> "when $\overline{\boldsymbol{\omega}^2}$ has increased to some value which depends on viscosity, it is no longer possible to neglect the effects of viscosity in the equation for the convervation of circulation."
>
> - G. I. Taylor & A. E. Green (1937)

In a laminar flow, these effects will became negligible in the limit as $\nu \to 0$, but in a turbulent flow they persist even in the inviscid limit.

The conclusion is that there is no length-scale in a turbulent flow at which circulations are conserved, for individual loops. In the inertial-range, the circulations are not conserved because of the subscale force terms $\mathbf{f}_\ell^{v,s}$. In the dissipation range, the circulations are not conserved because of the viscous force $\mathbf{f}_\nu^{\mathrm{vis}}$. At every length-scale, the circulations must be expected to change substantially in time and not to be conserved, as has often been assumed. This casts doubt on the standard arguments of Taylor, Tennekes & Lumley, etc. for growth of $\boldsymbol{\omega}^2(t)$, for positivity of $\omega_i S_{ij}\omega_j$, etc. which depend on conservation of circulations. Is there any way to see how these arguments might be somehow valid?

Recent fundamental progress on this question has come from an important mathematical result:

> P. Constantin & G. Iyer, "A stochastic Lagrangian representation of the three-dimensional incompressible Navier-Stokes equations," Commun. Pure Appl. Math. Vol. LXI, 03300345 (2008)

These authors have shown that there is a beautiful generalization of the Kelvin Theorem on conservation of circulations which applies to the incompressible Navier-Stokes equation. They show that circulations are conserved even for $\nu > 0$, not deterministically but in a precise stochastic sense! To state their result, one must consider an ensemble of *stochastic Lagrangian flows* $\widetilde{\mathbf{x}}(\tau|\mathbf{a},t)$, generated by SDE's

$$d\widetilde{\mathbf{x}} = \mathbf{u}(\widetilde{\mathbf{x}},\tau)d\tau + \sqrt{2\nu}d\widetilde{\mathbf{W}}(\tau), \quad \widetilde{\mathbf{x}}(t) = \mathbf{a} \tag{45}$$

where $\widetilde{\mathbf{W}}(\tau)$ is a d-dimensional vector Brownian motion. For a given velocity field $\mathbf{u}(\mathbf{x},t)$ in spacetime, this stochastic equation may be solved both forward and backward in time. Suppose that we consider a specific closed, rectifiable loop C at time t and generate the stochastic Lagrangian flows *backward in time.* Define the ensemble of loops generated by advecting this fixed loop C backward in time by

$$\widetilde{C}(\tau) = \widetilde{\mathbf{x}}(C,\tau), \quad \tau < t. \tag{46}$$

The figure below represents this infinite ensemble of loops with three typical samples:

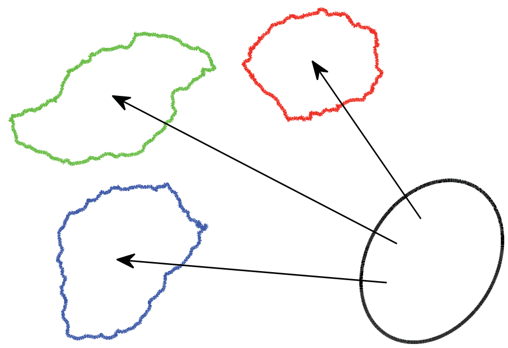


FIGURE. Shown are three members (red, green, and blue) of the infinite ensemble of loops obtained by stochastic advection of a loop C (black) at time t backward in time to $\tau < t$.

Constantin & Iyer (2008) have proved the following beautiful result: the spacetime velocity field $\mathbf{u}(\mathbf{x}, t)$ is a smooth solution of the incompressible Navier-Stokes equation if and only if

$$\oint_C d\mathbf{x} \cdot \mathbf{u}(\mathbf{x}, t) = \overline{\oint_{\widetilde{C}(\tau)} d\mathbf{x} \cdot \mathbf{u}(\mathbf{x}, \tau)} \tag{47}$$

for all rectifiable loops C and times $\tau < t$, where the overline $\overline{(\cdots)}$ denotes average over the ensemble of Brownian motions. Thus, circulations are statistically conserved backward in time[3]! Note that this result implies an "arrow of time." The statistical conservation law would hold forward in time instead for solutions of the negative-viscosity Navier-Stokes equation (which is well-posed only solved backward in time for given final conditions).

Formally, the stochastic Kelvin Theorem for incompressible Navier-Stokes equations reduces to the standard Kelvin Theorem for incompressible Euler equations in the limit as $\nu \to 0$. This is rigorously true if the Euler solution remains smooth in the limit. We shall return to the question how this conservation law behaves in high-Reynolds-number turbulent flow when we study turbulent Lagrangian dynamics in the next chapter.

[3] In probabilistic terminology, the stochastic process of circulations is a *backward martingale*.