(C) Orders of Magnitude & Dominant Balances

The orders of magnitude of the terms in the various balance equations can be worked out by the same techniques applied earlier for the velocity (momentum) and kinetic energy balances. For example,

$$
\begin{aligned}
\overline{D}_{\ell t}\bar{\boldsymbol{\omega}}_\ell &= (\partial_t + \bar{\mathbf{u}}\cdot\boldsymbol{\nabla})\bar{\boldsymbol{\omega}}_\ell \\
&= (\bar{\boldsymbol{\omega}}_\ell\cdot\boldsymbol{\nabla})\bar{\mathbf{u}}_\ell \;\; \}\; O(\frac{\delta u^2(\ell)}{\ell^2}) \\
&\quad +\boldsymbol{\nabla}\times\mathbf{f}^s_\ell \;\; \}\; O(\frac{\delta u^2(\ell)}{\ell^2}) \\
&\quad +\boldsymbol{\nabla}\times\bar{\mathbf{f}}_\ell \;\; \}\; O(||\boldsymbol{\nabla}\mathbf{f}||_\infty) \\
&\quad +\nu\triangle\bar{\boldsymbol{\omega}}_\ell \;\; \}\; O(\frac{\nu\delta u(\ell)}{\ell^2}) = O(\frac{\delta u^2(\ell)}{\ell^2}\cdot Re_\ell^{-1})
\end{aligned} \tag{19}
$$

The proofs are straightforward. For instance, using the "shift trick" twice, we see that

$$
\begin{aligned}
(\boldsymbol{\nabla}\times\mathbf{f}^s_\ell)_i &= \epsilon_{ijk}\cdot\frac{1}{\ell^2}\times\Big[\int d^3r\,(\partial_m\partial_k G)_\ell(\mathbf{r})\delta u_j(\mathbf{r})\delta u_m(\mathbf{r}) \\
&\quad -\int d^3r\,(\partial_m\partial_k G)_\ell(\mathbf{r})\delta u_m(\mathbf{r})\int d^3r'\,G_\ell(\mathbf{r}')\delta u_j(\mathbf{r}') \\
&\quad -\int d^3r(\partial_m G)_\ell(\mathbf{r})\delta u_j(\mathbf{r})\int d^3r'(\partial_k G)_\ell(\mathbf{r}')\delta u_m(\mathbf{r}')\Big] \\
&= O(\frac{\delta u^2(\ell)}{\ell^2})
\end{aligned} \tag{20}
$$

When $Re_\ell = \frac{\ell\delta u(\ell)}{\nu} \gg 1$, we see that the viscous diffusion term is negligible. Likewise, when

$$
\begin{aligned}
||\boldsymbol{\nabla}\mathbf{f}||_\infty^{1/2} &= \text{time rate of large-scale forcing} \\
&\ll \text{local eddy turnover-rate } = \frac{\delta u(\ell)}{\ell},
\end{aligned} \tag{21}
$$

then the external force term is negligible. Thus, the dominant terms are the vortex-stretching $(\bar{\boldsymbol{\omega}}_\ell\cdot\boldsymbol{\nabla})\bar{\mathbf{u}}_\ell$ and the subscale force term $\boldsymbol{\nabla}\times\mathbf{f}^s_\ell$, so that

$$\overline{D}_{\ell t}\bar{\boldsymbol{\omega}}_\ell = O(\tfrac{\delta u^2(\ell)}{\ell^2})$$

as well. This is as expected for the intrinsic Lagrangian evolution of the coarse-grained vorticity.

Similar estimates may be straightforwardly derived for the large-scale enstrophy balance

$$\begin{aligned} & \overbrace{\overline{D}_{\ell t}(\frac{1}{2}|\bar{\boldsymbol{\omega}}_\ell|^2)}^{O(\frac{\delta u^3(\ell)}{\ell^3})} + \overbrace{\boldsymbol{\nabla}\cdot(\bar{\boldsymbol{\omega}}_\ell \times \mathbf{f}_\ell^s)}^{O(\frac{\delta u^3(\ell)}{\ell^3})} - \overbrace{\nu\triangle(\frac{1}{2}|\bar{\boldsymbol{\omega}}_\ell|^2)}^{O(\frac{\delta u^3(\ell)}{\ell^3})\cdot Re_\ell^{-1}} \\ = & \underbrace{\bar{\boldsymbol{\omega}}_\ell^\top \overline{\mathbf{S}}_\ell \bar{\boldsymbol{\omega}}_\ell}_{O(\frac{\delta u^3(\ell)}{\ell^3})} + \underbrace{(\boldsymbol{\nabla}\times\bar{\boldsymbol{\omega}}_\ell)\cdot\mathbf{f}_\ell^s}_{O(\frac{\delta u^2(\ell)}{\ell^3})} - \underbrace{\nu|\boldsymbol{\nabla}\bar{\boldsymbol{\omega}}_\ell|^2}_{O(\frac{\delta u^3(\ell)}{\ell^3}\cdot Re_\ell^{-1})} + \underbrace{\bar{\boldsymbol{\omega}}_\ell\cdot(\boldsymbol{\nabla}\times\bar{\mathbf{f}}_\ell)}_{O(\frac{\delta u(\ell)}{\ell}||\boldsymbol{\nabla}\mathbf{f}||_\infty)} \end{aligned} \tag{22}$$

We see again that the external forcing and the viscous terms are all negligible relative to the inertial-range nonlinear terms.

In a steady-state, homogeneous turbulent state, we may also neglect the time-derivative and space-transport contributions. In that case, the dominant balance is

$$\langle \bar{\boldsymbol{\omega}}_\ell^\top \overline{\mathbf{S}}_\ell \bar{\boldsymbol{\omega}}_\ell \rangle \doteq -\langle (\boldsymbol{\nabla}\times\bar{\boldsymbol{\omega}}_\ell)\cdot\mathbf{f}_\ell^s \rangle,$$

so that the enstrophy produced by large-scale vortex-stretching is balanced by the transfer of enstrophy to the small-scales $< \ell$. This is quite different from the energy balance! First, the large-scale forcing plays no role, i.e. the enstrophy injected by the external force (if any) is negligible in the inertial-range. In fact, the vortex-stretching and the scale-transfer both rapidly increase for $\ell \ll L$. Within K41 theory, e.g., these grow as

$$\frac{\delta u^3(\ell)}{\ell^3} \sim \frac{\epsilon}{\ell^2} \to \infty \quad \text{as} \quad \ell \to 0$$

Intermittency effects do not change this basic picture, but simply modify the rate of divergence based on the local singularity strength. There is no constant flux of enstrophy to small-scales (i.e. independent of ℓ) because the transfer is fed by a growing rate of vortex-stretching.

Similar results should hold for other forms of the large-scale enstrophy balance, e.g. that given in T & L, eq.(3.3.36). However, they require delicate cancellations which cannot presently be proved rigorously to occur. For example, T&L, eq.(3.3.6) writes the enstrophy flux as

$$\bar{\omega}_{\ell i,j}\tau_\ell(\omega_i, u_j)$$

and has another stretching term

$$\bar{\omega}_{\ell i}\tau_\ell(\omega_j, S_{ij})$$

in addition to $\bar{\omega}_{\ell i}\overline{S}_{\ell ij}\bar{\omega}_{\ell j}$. Of those, the most dangerous is the second one, because both ω_j and S_{ij} are dissipation-range variables. A naive estimate in K41 theory would be that

$$\tau_\ell(\omega_j, S_{ij}) = O(\tfrac{\epsilon}{\nu})$$

which diverges as $Re \to \infty$. A better bound can be obtained by the observation

$$\tau_\ell(\omega_j, S_{ij}) = \partial_j \tau_\ell(\omega_j, u_i)$$

(cf. T&L, eq.(3.3.33)). We have already observed some time ago that

$$\tau_\ell(\omega_j, u_i) = O^*(\tfrac{\delta u^2(\ell)}{\ell})$$

because of decorrelation between the dissipation-range variable ω_j and the inertial-range variable u_i. A similar argument can be applied to $\partial_j \tau_\ell(\omega_j, u_i)$, together with the "shift trick" to move derivatives to the filter kernel. We thus obtain, heuristically,

$$\tau_\ell(\omega_j, S_{ij}) = \partial_j \tau_\ell(\omega_j, u_i) = O^*(\tfrac{\delta u^2(\ell)}{\ell^2}),$$

which is much smaller than the naive estimate $O(\epsilon/\nu)$. Altogether we then obtain that

$$\bar{\omega}_{\ell i} \tau_\ell(\omega_j, S_{ij}) = O^*(\tfrac{\delta u^3(\ell)}{\ell^3})$$

and

$$\bar{\omega}_{\ell i,j} \tau_\ell(\omega_j, u_j) = O^*(\tfrac{\delta u^3(\ell)}{\ell^3}).$$

These estimates are the same as these obtained before, but now much less rigorously.

We may likewise estimate the terms in the balance equation for the small-scale enstrophy ζ_ℓ. Note that this enstrophy itself is

$$\begin{aligned} \zeta_\ell &= \frac{1}{2}\tau_\ell(\omega_i, \omega_i) \\ &= \frac{1}{2}[\overline{(|\boldsymbol{\omega}|^2)_\ell} - |\bar{\boldsymbol{\omega}}_\ell|^2] \\ &= \frac{1}{2}\overline{(|\boldsymbol{\omega}|^2)_\ell} + O(\frac{\delta u^2(\ell}{\ell^2}) \end{aligned} \tag{23}$$

The first term is the dominant one, as can be seen by arguments like those for Kolomogorov's refined similarity hypothesis (RSH). In fact, the RSH can be applied directly, using the relation

$$\nu|\boldsymbol{\omega}|^2 = \underbrace{2\nu|\mathbf{S}|^2}_{\epsilon} - 2\nu\partial_i\partial_j(u_i u_j))$$

which implies

$$\begin{aligned} \overline{(\nu|\boldsymbol{\omega}|^2)_\ell} &= \bar{\epsilon}_\ell - \frac{2\nu}{\ell^2}\int d^d r\, (\partial_i\partial_j G)_\ell(\mathbf{r})\delta u_i(\mathbf{r})\delta u_j(\mathbf{r}) \\ &= \bar{\epsilon}_\ell + O(\nu\frac{\delta u^2(\ell)}{\ell^2}) \end{aligned}$$

$$\sim \quad (const.)\frac{\delta u^3(\ell)}{\ell} + O(\nu\frac{\delta u^2(\ell)}{\ell^2}) \text{ by RSH} \tag{24}$$

so that

$$\overline{(|\boldsymbol{\omega}|^2)_\ell} \sim (const.)\frac{\delta u^3(\ell)}{\nu\ell} + O(\frac{\delta u^2(\ell)}{\ell^2})$$

This estimates shows, incidentally, that an RSH based on enstrophy must lead to the same scaling as the RSH based on dissipation, asymptotically for high Reynolds number. This argues against the proposal of S. Chen et al., "Refined similarity hypothesis for transverse structure functions in fluid turbulence," Phys. Rev. Lett. **79** 2253-2256 (1997), except as a finite Reynolds number effect. Our estimate above for $\overline{(|\boldsymbol{\omega}|^2)_\ell}$ is analogous to that obtained by T&L, eq.(3.3.40), using K41 theory for ensemble average square vorticity:

$$\langle|\boldsymbol{\omega}|^2\rangle = O(\frac{u_{rms}^2}{\lambda^2}),$$

except that our result is modified to account for the effects of intermittency. To see the relationship, we may define a kind of "local Taylor scale"

$$\lambda^2(\ell) = \frac{\nu\ell}{\delta u(\ell)}$$

so that our estimate becomes

$$\overline{(|\boldsymbol{\omega}|^2)_\ell} = \frac{\delta u^2(\ell)}{\lambda^2(\ell)}$$

In the case of K41 scaling, $\delta u(\ell) \sim (\epsilon\ell)^{1/3}$, with $\ell = L$, our estimate then coincides with that of T&L. However, this rewriting of the estimate is purely cosmetic and we see no advantage in introducing the "hybrid" Taylor length-scale.

To estimate the other terms in the balance equation for the small-scale enstrophy ζ_ℓ we must develop a generalization of the RSH for pth-order products of the velocity-gradient, of the general type

$$\overline{(|\boldsymbol{\nabla}\mathbf{u}|^p)_\ell}.$$

Following the line of thinking of Kolmogorov's original RSH, we may postutate that [2]

$$\overline{(|\boldsymbol{\nabla}\mathbf{u}|^p)_\ell} = |\frac{\delta u(\ell)}{\ell}|^p\, F_p(Re_\ell), \quad 0 < \ell < L \quad \text{(GRSH)}.$$

[2] This hypothesis is closely related to the "reverse RSH" proposed by S. Chen et al., Phys. Rev. Lett. **74** 1755-1758 (1995).

where F_p is a universal function of the local Reynolds number

$$Re_\ell = \frac{\ell\delta u(\ell)}{\nu}.$$

This conjecture is reasonable in both of the limits $\ell \to L$ and $\ell \to 0$. In the first limit $\ell \to L$, using $\delta u(L) \cong u_{rms}$, one gets that

$$\begin{aligned} \langle|\mathbf{\nabla u}|^p\rangle &\cong \overline{(|\mathbf{\nabla u}|^p)_L} \\ &= |\frac{\delta u(L)}{L}|^p\, F_p(Re_L) \cong |\frac{u_{rms}}{L}|^p\, F_p(Re) \end{aligned} \tag{25}$$

since $Re_L \cong Lu_{rms}/\nu = Re$. From this limit we may identify the function F_p as the same one that appears in the ensemble averages of velocity-gradients, $\langle|\mathbf{\nabla u}|^p\rangle = |u_{rms}/L|^p\, F_p(Re)$. In the other limit $\ell \to 0$, since both

$$\overline{(|\mathbf{\nabla u}|^p)_\ell} \to |\mathbf{\nabla u}|^p, \quad |\frac{\delta u(\ell)}{\ell}|^p \to |\mathbf{\nabla u}|^p$$

we just get that $F_p(0) = 1$. Here we have assumed that $\ell\delta u(\ell) \to 0$ as $\ell \to 0$ or, equivalently, $h_{min} > -1$ in the multifractal phenomenology.

In the opposite limit, $Re_\ell \gg 1$, we get from the GRSH that

$$\overline{(|\mathbf{\nabla u}|^p)_\ell} \sim |\frac{\delta u(\ell)}{\ell}|^p\, Re_\ell^{\gamma_p}$$

with

$$\gamma_p = \sup_h[\frac{(1-h)p - \kappa(h)}{1+h}] \tag{26}$$

in the multifractal formalism. This relation is rendered plausible by the observation that the singularity sets $\mathcal{S}(h)$ for a Hölder exponent h are generally dense in space for multifractal functions (Jaffard, 1997). Thus, the same ideas applied to the global average over space should also be applicable to the local averages over space-regions of diameter $\sim \ell$.

It should be interesting to check the above ideas by simulation and experiment. However, we shall use them below just to motivate some general plausible estimates that should be likely to be true, even if the above specific conjectures are not confirmed. The most important such plausible result is this:

$\overline{(|\mathbf{\nabla u}|^p)_\ell}$ should diverge as $Re_\ell \to \infty$, more rapidly for larger p.

To argue for this, note that $\gamma_p > 0$ for a specific p only if the h_p at which supremum in (**??**) is achieved satisfies

$$h_p < 1.$$

Therefore, if $\gamma_p > 0$ for all p, then it must also be true from formula (**??**) that

$$\gamma_p \nearrow \quad \text{as} \quad p \nearrow,$$

i.e. γ_p is increasing in p. In particular, we shall apply this result below to estimate the relative sizes of the local averages $\overline{(|\boldsymbol{\nabla}\mathbf{u}|^p)_\ell}$ for ℓ fixed and $\nu \to 0$.

We now estimate the various terms in the balance equation for ζ_ℓ. For simplicity, we consider the equation in the same form as T&L, eq.(3.3.38):

$$\begin{aligned} &\overline{D}_{\ell t}\zeta_\ell + \frac{1}{2}\partial_k \tau_\ell(\omega_i,\omega_i,u_k) - \nu\partial_k^2\zeta_\ell \\ &\quad = \tau_\ell(S_{ij},\omega_i,\omega_j) + \tau_\ell(\omega_i,\omega_j)\overline{S}_{\ell ij} + \tau_\ell(\omega_i,S_{ij})\bar{\omega}_{\ell j} \\ &\quad -\bar{\omega}_{\ell i,k}\tau_\ell(\omega_i,u_k) - \nu\tau_\ell(\omega_{i,k},\omega_{i,k}) \end{aligned} \tag{27}$$

Of course, the most important is

$$\tau_\ell(S_{ij},\omega_i,\omega_j) = \overline{(S_{ij}\omega_i\omega_j)_\ell} - \overline{S}_{\ell ij}\overline{(\omega_i\omega_j)_\ell} - 2\bar{\omega}_{\ell i}\overline{(S_{ij}\omega_j)_\ell} + 2\overline{S}_{\ell ij}\bar{\omega}_{\ell i}\bar{\omega}_{\ell j}$$

We can apply the previous estimates on $\overline{(|\boldsymbol{\nabla}\mathbf{u}|^p)_\ell}$ to obtain

$$\begin{aligned} \overline{S}_{\ell ij}\bar{\omega}_{\ell i}\bar{\omega}_{\ell j} &= O(\frac{\delta u^3(\ell)}{\ell^3}) \\ \overline{S}_{\ell ij}\overline{(\omega_i\omega_j)_\ell} &= O(\frac{\delta u(\ell)}{\ell}\overline{(|\boldsymbol{\nabla}\mathbf{u}|^2)_\ell}) \\ &= O^*(\frac{\delta u(\ell)}{\ell}\cdot\frac{\delta u^3(\ell)}{\nu\ell}) = O^*(\frac{\delta u^3(\ell)}{\ell^3}\cdot Re_\ell) \end{aligned} \tag{28}$$

Similarly

$$\bar{\omega}_{\ell i}\overline{(S_{ij}\omega_j)_\ell} = O^*(\tfrac{\delta u^3(\ell)}{\ell^3}\cdot Re_\ell)$$

and

$$\overline{(S_{ij}\omega_i\omega_j)_\ell} = O(\overline{(|\boldsymbol{\nabla}\mathbf{u}|^3)_\ell}) = O^*(\tfrac{\delta u^3(\ell)}{\ell^3}\cdot Re_\ell^{\gamma_3})$$

Because $\gamma_3 > \gamma_2 = 1$, we can see that the final term is the dominant one, so that for $Re_\ell \gg 1$

$$\tau_\ell(S_{ij},\omega_i,\omega_j) \cong \overline{(S_{ij}\omega_i\omega_j)_\ell} = O^*(\frac{\delta u^3(\ell)}{\ell^3}\cdot Re_\ell^{\gamma_3}) \tag{29}$$

The other terms that appear in the balance equation for the small-scale enstrophy are more easily estimated. Applying the various methods developed earlier, that should now be quite standard, we obtain

$$\begin{aligned}
\overline{D}_{\ell t}\zeta_\ell &= O^*(\frac{\delta u(\ell)}{\ell}\cdot\frac{\delta u^3(\ell)}{\nu\ell}) = O^*(\frac{\delta u^3(\ell)}{\ell^3}\cdot Re_\ell) \\
\partial_k\tau_\ell(\omega_i,\omega_i,u_k) &= O^*(\frac{1}{\ell}\cdot\frac{\delta u^3(\ell)}{\nu\ell}\cdot\delta u(\ell)) = O^*(\frac{\delta u^3(\ell)}{\ell^3}\cdot Re_\ell) \\
\nu\partial_k^2\zeta_\ell &= O^*(\nu\cdot\frac{1}{\ell^2}\cdot\frac{\delta u^3(\ell)}{\nu\ell}) = O^*(\frac{\delta u^3(\ell)}{\ell^3}) \\
\tau_\ell(S_{ij},\omega_i,\omega_j) &= O^*(\frac{\delta u^3(\ell)}{\ell^3}\cdot Re_\ell^{\gamma_3}) \\
\tau_\ell(\omega_i,\omega_j)\overline{S}_{\ell ij} &= O^*(\frac{\delta u^3(\ell)}{\nu\ell}\cdot\frac{\delta u(\ell)}{\ell}) = O^*(\frac{\delta u^3(\ell)}{\ell^3}\cdot Re_\ell) \\
\tau_\ell(\omega_i,S_{ij})\bar{\omega}_{\ell j} &= O^*(\frac{\delta u^2(\ell)}{\ell^2}\cdot\frac{\delta u(\ell)}{\ell}) = O^*(\frac{\delta u^3(\ell)}{\ell^3}) \\
\bar{\omega}_{\ell i,k}\tau_\ell(\omega_i,u_k) &= O^*(\frac{\delta u(\ell)}{\ell^2}\cdot\frac{\delta u^2(\ell)}{\ell}) = O^*(\frac{\delta u^3(\ell)}{\ell^3}) \\
\nu\tau_\ell(\omega_{i,k},\omega_{i,k}) &= ????
\end{aligned} \tag{30}$$

The above estimates should be compared with these of T&L, eqs.(3.3.54-61). Since $\gamma_3 > \gamma_2 = 1$, the largest of the above terms is $\tau_\ell(S_{ij},\omega_i,\omega_j)$ for $Re_\ell \gg 1$. Thus, only the last, unestimated term can balance it. We conclude that

$$\tau_\ell(S_{ij},\omega_i,\omega_j) \cong \nu\tau_\ell(\omega_{i,k},\omega_{i,k}) = O^*(\tfrac{\delta u^3(\ell)}{\ell^3}Re^{\gamma_3}) \quad \text{for} \quad Re_\ell \gg 1.$$

In fact, we can give an independent argument for this conclusion, employing the multifractal formalism. Note that

$$\begin{aligned}
\nu\tau_\ell(\omega_{i,k},\omega_{i,k}) &= \nu\overline{(|\boldsymbol{\nabla}\boldsymbol{\omega}|^2)_\ell} - \nu|\boldsymbol{\nabla}\bar{\boldsymbol{\omega}}_\ell|^2 \\
&\cong \nu\overline{(|\boldsymbol{\nabla}\boldsymbol{\omega}|^2)_\ell} \\
&= O(\nu\overline{(|\boldsymbol{\nabla}^2\mathbf{u}|^2)_\ell})
\end{aligned} \tag{31}$$

If we estimate $|\boldsymbol{\nabla}^2\mathbf{u}| \sim \delta u(\eta_h)/\eta_h^2 \sim u_0(\eta_h/L)^h/\eta_h^2 \sim \frac{u_0}{L^2}Re^{\frac{2-h}{1+h}}$ locally at a point with Hölder exponent h, then the GRSH should give

$$\overline{(|\boldsymbol{\nabla}^2\mathbf{u}|^2)_\ell} = O^*(\tfrac{\delta u^2(\ell)}{\ell^4}Re_\ell^{\gamma_2^{(2)}})$$

with

$$\gamma_2^{(2)} = \sup_h [\frac{(2-h)2-\kappa(h)}{1+h}]$$

so that

$$\nu\overline{(|\nabla^2\mathbf{u}|^2)_\ell} = O^*(\frac{\delta u^3(\ell)}{\ell^3}\cdot\frac{\nu}{\ell\delta u(\ell)}Re_\ell^{\gamma_2^{(2)}}) = O^*(\frac{\delta u^3(\ell)}{\ell^3}Re_\ell^{\gamma_2^{(2)}-1}).$$

However,

$$\gamma_2^{(2)} - 1 = \sup_h[\frac{(2-h)2-\kappa(h)}{1+h} - 1] = \sup_h[\frac{(1-h)3-\kappa(h)}{1+h}] = \gamma_3!$$

This reproduces the estimate that we obtained above.

The conclusion of all these considerations is that the dominant balance in the equation for small-scale enstrophy is between the small-scale vortex-stretching and the viscous destruction:

$$\tau_\ell(S_{ij},\omega_i,\omega_j) \cong \nu\tau_\ell(\omega_{i,j},\omega_{i,j})$$

for $Re_\ell \gg 1$. This should be compared with the dominant balance in the equation for the large-scale enstrophy, which we showed earlier is between the large-scale vortex-stretching and the nonlinear transfer to small scales (or the "subscale dissipation" of enstrophy.) Thus, we see that the dynamics of enstrophy is not cascade-like. Although the flux of enstrophy to smaller scales mainly absorbs the production by large-scale stretching, that transfer plays very little role in the balance of the small-scale enstrophy. At every scale, the vortex-stretching at that scale plays a dominant role, balanced by the largest dissipation mechanism available (subscale force or viscous force).

The above conclusion was already reached by G. I. Taylor, "Production and dissipation of vorticity in a turbulent fluid," Proc. Roy. Soc. Lond. A **164** 15-23(1938). By considering the balance equation for the fine-grained enstrophy

$$\partial_t(\tfrac{1}{2}|\boldsymbol{\omega}|^2) + \nabla\cdot[\tfrac{1}{2}|\boldsymbol{\omega}|^2\mathbf{u} - \nu\nabla(\tfrac{1}{2}|\boldsymbol{\omega}|^2)] = \boldsymbol{\omega}^\top\mathbf{S}\boldsymbol{\omega} - \nu|\nabla\boldsymbol{\omega}|^2,$$

he argued that

$$\langle\boldsymbol{\omega}^\top\mathbf{S}\boldsymbol{\omega}\rangle \;\cong\; \langle\nu|\nabla\boldsymbol{\omega}|^2\rangle$$

when $Re \gg 1$, even for turbulence that may be neither homogeneous nor stationary. The corresponding balance equation for the large-scale enstrophy of the coarse-grained vorticity has been considered by C. Meneveau, "Statistics of turbulence subgrid-scale stresses: necessary conditions and experimental tests," Phys. Fluids **6** 815-833 (1994). The discussion of the balance equation for subscale enstrophy ζ_ℓ that we have given here appears to be new in its details, but agrees with the earlier analyses.

An important side-product of the above considerations is that (small-scale) vortex-stretching should be positive on average. In fact,

$$\nu|\boldsymbol{\nabla}\boldsymbol{\omega}|^2 \geq 0 \implies \langle \boldsymbol{\omega}^\top \mathbf{S}\boldsymbol{\omega}\rangle \;\cong\; \langle \nu|\boldsymbol{\nabla}\boldsymbol{\omega}|^2\rangle \;\geq 0$$

for $Re \gg 1$. Likewise, the local averages over regions of size ℓ should satisfy

$$\tau_\ell(S_{ij},\omega_i,\omega_j) \cong \nu\tau_\ell(\omega_{i,j},\omega_{i,j}) \geq 0$$

for $Re_\ell \gg 1$, or, equivalently, $\overline{(S_{ij}\omega_i\omega_j)_\ell} \cong \nu\overline{(|\omega_{i,j}|^2)_\ell} \geq 0$. Of course, pointwise the quantity $S_{ij}\omega_i\omega_j$ alternates in sign, taking values both positive and negative. However, on average, vortex-stretching must exceed vortex-contraction. It is important to note that we have not given an *a priori* proof of this, but simply shown that it emerges as a consistency condition. It would be very important to give a deeper dynamical justification of the positivity of the vortex-stretching term, a question to which we shall turn in the following subsection. The property of vortex-stretching is often considered to be so essential to turbulence that the entire phenomenon of turbulence is defined by that feature. For example, T & L write that

> "In two-dimensional 'turbulence' there is no vortex stretching, so that the vorticity budget (3.3.62) is irrelevant in that case. This implies that the spectral energy-transfer concepts developed here do not apply to two-dimensional stochastic flow fields." —Tennekes & Lumley, p.91.

As a matter of fact, this strict identification of turbulence with vortex-stretching is now generally regarded as too restrictive. It was pointed out by

> R. H. Kraichnan, "Inertial ranges in two-dimensional turbulence," Phys. Fluids **10** 1417-1423(1967)

that there is a range of spectral energy transfer in 2D, although the transfer is an inverse cascade from small-scales to large-scales! There is also a dual cascade of enstrophy from large-scales to small-scales in 2D. It was furthermore emphasized by

> G. K. Batchelor, "Computation of the energy spectrum in homogeneous, two-dimensional turbulence," Phys. Fluids **12** II 233-239(1969)

that, while there is no vortex-stretching in 2D, there is stretching of vorticity-gradients. We shall return to this topic of two-dimensional turbulence in a later chapter!