(B) Dynamics of the Coarse-Grained Vorticity

See T & L, Section 3.3

Since our interest is mainly in inertial-range turbulence dynamics, we shall consider first the equations of the coarse-grained vorticity. However, it pays to consider a somewhat more general problem of the dynamics of coarse-grained velocity-gradients. Taking as the starting point the equation

$$\partial_t \bar{\mathbf{u}}_\ell + (\bar{\mathbf{u}}_\ell \cdot \boldsymbol{\nabla})\bar{\mathbf{u}}_\ell = \mathbf{f}^s_\ell - \boldsymbol{\nabla}\bar{p}_\ell + \nu\triangle\bar{\mathbf{u}}_\ell + \bar{\mathbf{f}}_\ell$$

for the coarse-grained velocity, one easily derives, by taking the gradient $\boldsymbol{\nabla}$ with respect to $\mathbf{x}$, that

$$\overline{D}_t\overline{\mathbf{A}}_\ell = -\overline{\mathbf{A}}^2_\ell + \overline{\mathbf{H}}_\ell + \mathbf{F}^s_\ell + \nu\triangle\overline{\mathbf{A}}_\ell + \overline{\mathbf{F}}_\ell$$

where

$$\begin{aligned} A_{ij} \equiv \frac{\partial u_i}{\partial x_j} &= \text{velocity-gradient tensor} \\ H_{ij} = \frac{\partial^2 p}{\partial x_i \partial x_j} &= \text{pressure-tension} \\ \overline{D}_{\ell,t} = \partial_t + \bar{\mathbf{u}}_\ell \cdot \boldsymbol{\nabla}_x &= \text{material derivative of coarse-grained velocity} \\ (F^s_\ell)_{ij} = \frac{\partial f^s_{\ell i}}{\partial x_j} &= \text{subscale force-gradient} \\ F_{ij} = \frac{\partial f_i}{\partial x_j} &= \text{external force-gradient} \end{aligned} \tag{7}$$

A further simplification can be made if one uses the trace-free condition, $\mathrm{tr}(\overline{\mathbf{A}}_\ell) = 0$, which implies that

$$-\mathrm{tr}(\overline{\mathbf{A}}^2_\ell) + \mathrm{tr}(\overline{\mathbf{H}}_\ell) + \mathrm{tr}(\mathbf{F}^s_\ell) + \mathrm{tr}(\overline{\mathbf{F}}_\ell) = 0$$

Multiplying by $1/d$ and subtracting from the previous equation gives

$$\boxed{\overline{D}_t\overline{\mathbf{A}}_\ell = -\left(\overline{\mathbf{A}}^2_\ell - \frac{1}{d}\mathrm{tr}(\overline{\mathbf{A}}^2_\ell)\mathbf{I}\right) + \mathring{\overline{\mathbf{H}}}_\ell + \mathring{\mathbf{F}}^s_\ell + \nu\triangle\overline{\mathbf{A}}_\ell + \mathring{\overline{\mathbf{F}}}_\ell \qquad (8)}$$

where the notation $\mathring{\mathbf{M}}$ for a $d \times d$ square matrix $\mathbf{M}$ denotes in general the trace-free or deviatoric part of $\mathbf{M}$, i.e.

$$\overset{\circ}{\mathbf{M}} = \mathbf{M} - \frac{1}{d}\mathrm{tr}(\mathbf{M})\mathbf{I}$$

There is a considerable history of phenomenological closure modelling of equation (8), going back to the seminal work of

> P. Vieillefosse, "Local interaction between vorticity and shear in a perfect incompressible flow," J. Physique (Paris) **43** 837-842(1982); "Internal motion of a small element of fluid in an inviscid flow," Physica **A 125** 150-162(1984)

who introduced the simplest approximation of setting $\overset{\circ}{\overline{\mathbf{H}}}_\ell = \overset{\circ}{\mathbf{F}}{}^s_\ell = \nu\triangle\overline{\mathbf{A}}_\ell = \overset{\circ}{\overline{\mathbf{F}}}_\ell = 0$, the so-called <u>Vieillefosse model</u> or <u>restricted Euler model</u>

$$\overline{D}_t\overline{\mathbf{A}}_\ell = -(\overline{\mathbf{A}}_\ell^2 - \frac{1}{d}\mathrm{tr}(\overline{\mathbf{A}}_\ell^2)\mathbf{I}).$$

This model has been much studied subsequently, notably by

> B. J. Cantwell, "Exact solution of a restricted Euler equation for the velocity gradient," Phys. Fluids A **A** 782-793(1992)

who obtained an exact solution in terms of elliptic functions. Many further refinements of this approach, with models for $\nu\triangle\overline{\mathbf{A}}_\ell, \overset{\circ}{\overline{\mathbf{H}}}_\ell$ and (for $\ell \neq 0$) $\overset{\circ}{\mathbf{F}}{}^s_\ell$ have been proposed. See

> C. Meneveau, "Lagrangian dynamics and models of the velocity gradient tensor in turbulent flows," Annu. Rev. Fluid Mech. **43** 219-245 (2011)

for recent results and a rather complete bibliography of the literature. One of the interesting results of statistical models of this type is that they naturally predict that

$$\langle \bar{\boldsymbol{\omega}}_\ell^\top \overline{\mathbf{S}}_\ell \bar{\boldsymbol{\omega}}_\ell \rangle \ > \ 0$$

and more refined statistical alignment trends of turbulent velocity gradients, such as the preferential alignment of $\bar{\boldsymbol{\omega}}_\ell$ with the intermediate strain direction, as observed in DNS by

> W. Ashurst, A. Kerstein, R. Kerr and C. Gibson, "Alignment of vorticity and scalar gradient with strain rate in simulated Navier-Stokes turbulence," Phys. Fluids **30** 2343-2353(1987)

Thus it seems that we have an explanation of the fundamental observation that $\langle \bar{\boldsymbol{\omega}}_\ell^\top \overline{\mathbf{S}}_\ell \bar{\boldsymbol{\omega}}_\ell \rangle > 0$! The difficulty is that all of these models make fairly crude and hand-waving approximations of the unclosed terms, in particular the pressure-Hessian $\mathring{\overline{\mathbf{H}}}_\ell$. While the models seem to make remarkably good predictions for the statistics of $\overline{\mathbf{A}}_\ell$, it is not clear that the modelling approximations yield accurate enough results for $\mathring{\overline{\mathbf{H}}}_\ell$ itself (and other unclosed terms). The good news is that these models suggest that much of the observed features of $\overline{\mathbf{A}}_\ell$ can be understood from the <u>local self-stretching</u> dynamics contained in the closed term $-\overline{\mathbf{A}}_\ell^2$.

The equation for the velocity-gradient can be separated into equations for the symmetric part

$$\overline{D}_t \overline{\mathbf{S}}_\ell = -[\overline{\mathbf{S}}_\ell^2 + \overline{\boldsymbol{\Omega}}_\ell^2 - \tfrac{1}{d}\mathrm{tr}(\overline{\mathbf{S}}_\ell^2 + \overline{\boldsymbol{\Omega}}_\ell^2)] + \mathring{\overline{\mathbf{H}}}_\ell + \tfrac{1}{2}(\mathring{\mathbf{F}}_\ell^s + \mathring{\mathbf{F}}_\ell^{sT}) + \tfrac{1}{2}(\overline{\mathbf{F}}_\ell + \overline{\mathbf{F}}_\ell{}^\top) + \nu \triangle \overline{\mathbf{S}}_\ell.$$

and anti-symmetric part

$$\overline{D}_t \overline{\boldsymbol{\Omega}}_\ell = -(\overline{\mathbf{S}}_\ell \overline{\boldsymbol{\Omega}}_\ell + \overline{\boldsymbol{\Omega}}_\ell \overline{\mathbf{S}}_\ell) + \tfrac{1}{2}(\mathring{\mathbf{F}}_\ell^s - \mathring{\mathbf{F}}_\ell^{sT}) + \tfrac{1}{2}(\overline{\mathbf{F}}_\ell - \overline{\mathbf{F}}_\ell{}^\top) + \nu \triangle \overline{\boldsymbol{\Omega}}_\ell.$$

Although these equations are superficially similar, there are profound differences. For example, note that the (nasty) pressure-Hessian term is absent from the equation for $\overline{\boldsymbol{\Omega}}_\ell$. In fact, the latter equation is mathematically equivalent to the equation for $\bar{\mathbf{u}}_\ell$! It is more commonly written in terms of the <u>filtered vorticity vector</u> $\bar{\boldsymbol{\omega}}_\ell$, using

$$\Omega_{ij} = -\tfrac{1}{2}\epsilon_{ijk}\omega_k, \quad \omega_i = -\epsilon_{ijk}\Omega_{jk},$$

which gives

$$\overline{D}_t \bar{\boldsymbol{\omega}}_\ell = \overline{\mathbf{S}}_\ell \bar{\boldsymbol{\omega}}_\ell + \boldsymbol{\nabla} \times (\mathbf{f}_\ell^s + \bar{\mathbf{f}}_\ell) + \nu \triangle \bar{\boldsymbol{\omega}}_\ell,$$

or

$$(\partial_t + \bar{\mathbf{u}}_\ell \cdot \boldsymbol{\nabla})\bar{\boldsymbol{\omega}}_\ell = (\bar{\boldsymbol{\omega}}_\ell \cdot \boldsymbol{\nabla})\bar{\mathbf{u}}_\ell + \boldsymbol{\nabla} \times (\mathbf{f}_\ell^s + \bar{\mathbf{f}}_\ell) + \nu \triangle \bar{\boldsymbol{\omega}}_\ell \ \text{(using } \overline{\boldsymbol{\Omega}}_\ell \bar{\boldsymbol{\omega}}_\ell = 0!\text{)} \tag{9}$$

or

$$\partial_t \bar{\boldsymbol{\omega}}_\ell = \boldsymbol{\nabla} \times (\bar{\mathbf{u}}_\ell \times \bar{\boldsymbol{\omega}}_\ell + \mathbf{f}_\ell^s + \bar{\mathbf{f}}_\ell - \nu \boldsymbol{\nabla} \times \bar{\boldsymbol{\omega}}_\ell) \ \text{(using vector calculus identities)} \tag{10}$$

An important feature of this equation is that it has the general conservation form

$$\partial_t(\bar{\boldsymbol{\omega}}_\ell)_i + \partial_j(\overline{\boldsymbol{\Sigma}}_\ell)_{ji} = 0 \tag{11}$$

where $(\overline{\boldsymbol{\Sigma}}_\ell)_{ji}$ = space flux of the ith-component of vorticity in the jth direction and is an anti-symmetric matric

$$\begin{aligned}(\overline{\boldsymbol{\Sigma}}_\ell)_{ij} = {} & \underbrace{\bar{u}_{\ell i}\bar{\omega}_{\ell j}}_{\text{advection of vorticity}} - \underbrace{\bar{u}_{\ell j}\bar{\omega}_{\ell i}}_{\text{stretching of vorticity}} \\ & + \underbrace{\nu\left(\frac{\partial\bar{\omega}_{\ell i}}{\partial x_j} - \frac{\partial\bar{\omega}_{\ell j}}{\partial x_i}\right)}_{\text{viscous diffusion of vorticity}} \\ & + \epsilon_{ijk}\Big(\underbrace{\mathbf{f}^s_\ell}_{\text{transport of vorticity by turbulent subscale force}} \\ & + \underbrace{\bar{\mathbf{f}}_\ell}_{\text{transport of vorticity by external force}}\Big)_k \end{aligned} \tag{12}$$

Note that we can replace $\epsilon_{ijk}\mathbf{f}^s_{\ell k}$ with the contribution from the turbulent vortex force

$$\epsilon_{ijk}\mathbf{f}^v_{\ell k} = \tau_\ell(u_i, \omega_j) - \tau_\ell(u_j, \omega_i)$$

since the difference, $\epsilon_{ijk}(f^s_{\ell k} - f^v_{\ell k}) = -\epsilon_{ijk}\partial_k k_\ell$, does not contribute to the divergence $\partial_j(\overline{\boldsymbol{\Sigma}}_\ell)_{ji}$. Thus, we see that the turbulent term comes from subscale vorticity advection and stretching.

Although (11) has the "form" of a conservation law, it is not a rather unconventional conservation law since, generally,

$$\textstyle\int_V \boldsymbol{\omega} d^3x = \mathbf{0}!$$

Indeed, using $\partial_j(x_i\omega_j) = \delta_{ij}\omega_j + \underbrace{x_i(\partial_j\omega_j)}_{=0} = \omega_i$, one can see that

$$\textstyle\int_V \boldsymbol{\omega} d^3x = \int_{\partial V} \mathbf{x}(\boldsymbol{\omega}\cdot\hat{\mathbf{n}})d\mathbf{A} = \mathbf{0},$$

if $\boldsymbol{\omega}\cdot\hat{\mathbf{n}} = 0$ at the surface ∂V. This will be true if $\boldsymbol{\omega}$ is a compactly supported vorticity distribution away from any boundaries. It will also be true in the presence of boundaries, since stick boundary conditions on the velocity ($\mathbf{u} = \mathbf{0}$) imply that $\boldsymbol{\omega}\cdot\hat{\mathbf{n}} = 0$, i.e. the normal component of $\boldsymbol{\omega}$ at the boundary vanishes. Thus, the "conserved" quantity $\int_V \boldsymbol{\omega} d^3x$ is trivial!

A nontrivial quantity is the enstrophy $\Omega = \frac{1}{2}\int_V |\boldsymbol{\omega}|^2 d^3x$. The local balance equation for large-scale enstrophy can be obtained from (10), as:

$$\boxed{\begin{aligned} &\partial_t(\frac{1}{2}|\bar{\boldsymbol{\omega}}_\ell|^2) + \boldsymbol{\nabla}\cdot[\overbrace{\frac{1}{2}|\bar{\boldsymbol{\omega}}_\ell|^2\bar{\mathbf{u}}_\ell}^{\text{large-scale advection}} + \overbrace{\bar{\boldsymbol{\omega}}_\ell\times\mathbf{f}_\ell^s}^{\text{turbulent diffusion}} - \overbrace{\nu\boldsymbol{\nabla}(\frac{1}{2}|\bar{\boldsymbol{\omega}}_\ell|^2)}^{\text{viscous diffusion}}] \\ = &\ \bar{\boldsymbol{\omega}}_\ell^\top\overline{\mathbf{S}}_\ell\bar{\boldsymbol{\omega}}_\ell \rightsquigarrow \text{production of large-scale enstrophy by vortex-stretching} \\ + &\ (\boldsymbol{\nabla}\times\bar{\boldsymbol{\omega}}_\ell)\cdot\mathbf{f}_\ell^s \rightsquigarrow \text{flux of large-scale enstrophy to the small-scales} \\ - &\ \nu|\boldsymbol{\nabla}\bar{\boldsymbol{\omega}}_\ell|^2 \rightsquigarrow \text{viscous dissipation of large-scale enstrophy} \\ + &\ \bar{\boldsymbol{\omega}}_\ell\cdot(\boldsymbol{\nabla}\times\bar{\mathbf{f}}_\ell) \rightsquigarrow \text{production of large-scale enstrophy by external force} \qquad (13) \end{aligned}}$$

This does not have the conservation form — even formally — in the limit as $\nu \to 0$ and $\ell \to 0$, because of the production by vortex-stretching. This balance equation can be written in other, equivalent forms. For example, replacing $\mathbf{f}_\ell^s \to \mathbf{f}_\ell^v$ gives

$$\begin{aligned} \partial_t(\frac{1}{2}|\bar{\boldsymbol{\omega}}_\ell|^2) + \partial_k[\frac{1}{2}|\bar{\boldsymbol{\omega}}_\ell|^2\bar{u}_{\ell k} + \bar{\omega}_{\ell i}(\tau_\ell(\omega_i,u_k) - \tau_\ell(\omega_k,u_i)) - \nu\partial_k(\frac{1}{2}|\bar{\boldsymbol{\omega}}_\ell|^2)] \\ = \bar{\boldsymbol{\omega}}_\ell^\top\overline{\mathbf{S}}_\ell\bar{\boldsymbol{\omega}}_\ell + \bar{\omega}_{\ell j,k}(\tau_\ell(\omega_j,u_k) - \tau_\ell(\omega_k,u_j)) - \nu|\boldsymbol{\nabla}\bar{\boldsymbol{\omega}}_\ell|^2 \end{aligned} \tag{14}$$

Using the relations

$$\partial_j\tau_\ell(\omega_j,u_i) = \tau_\ell(\omega_j,u_{i,j}) = \tau_\ell(\omega_j,S_{ij})$$

this can also be written as

$$\begin{aligned} \partial_t(\frac{1}{2}|\bar{\boldsymbol{\omega}}_\ell|^2) + \partial_k[\frac{1}{2}|\bar{\boldsymbol{\omega}}_\ell|^2\bar{u}_{\ell k} + \bar{\omega}_{\ell i}\tau_\ell(\omega_i,u_k) - \nu\partial_k(\frac{1}{2}|\bar{\boldsymbol{\omega}}_\ell|^2)] \\ = \bar{\boldsymbol{\omega}}_\ell^\top\overline{\mathbf{S}}_\ell\bar{\boldsymbol{\omega}}_\ell + \bar{\omega}_{\ell i}\tau_\ell(\omega_j,S_{ij}) + \bar{\omega}_{\ell i,j}\tau_\ell(\omega_i,u_j) - \nu|\boldsymbol{\nabla}\bar{\boldsymbol{\omega}}_\ell|^2 \end{aligned} \tag{15}$$

This is the form considered by T & L, eq. (3.3.36).

A corresponding equation can be written for the small-scale enstrophy $\zeta_\ell = \frac{1}{2}\tau_\ell(\omega_i,\omega_i)$ which takes the form

$$
\begin{aligned}
&\partial_t\zeta_\ell + \\
&\partial_k\Big[\overbrace{\zeta_\ell\bar{u}_{\ell k}}^{\text{large-scale advection}} + \overbrace{\frac{1}{2}\tau_\ell(\omega_i,\omega_i,u_k)}^{\text{small-scale advection}} + \overbrace{\bar{\omega}_{\ell i}\tau_\ell(\omega_k,u_i)}^{\text{turbulent diffusion}} - \overbrace{\nu\partial_k\zeta_\ell}^{\text{viscous diffusion}}\Big] \\
&\quad = \underbrace{\tau_\ell(S_{ij},\omega_i,\omega_j) + \tau_\ell(\omega_i,\omega_j)\overline{S}_{ij} + 2\tau_\ell(\omega_i,S_{ij})\bar{\omega}_j}_{\text{vortex stretching involving small-scales}} \\
&\quad - \underbrace{\bar{\omega}_{\ell i,k}(\tau_\ell(\omega_i,u_k) - \tau_\ell(\omega_k,u_i))}_{\text{flux of enstrophy from large-scales}} - \underbrace{\nu\tau_\ell(\omega_{i,k},\omega_{i,k})}_{\text{viscous destruction of enstrophy}} \qquad (16)
\end{aligned}
$$

The term we called "turbulent diffusion" of the small-scale enstrophy should really be written

$$
\bar{\omega}_{\ell i}\tau_\ell(\omega_k,u_i) = \underbrace{\bar{\omega}_{\ell i}\tau_\ell(\omega_i,u_k)}_{\text{small-scale advection contribution}} + \underbrace{\bar{\omega}_{\ell i}(\tau_\ell(\omega_k,u_i) - \tau_\ell(\omega_i,u_k))}_{\text{turbulent diffusion proper}} \qquad (17)
$$

Where the vortex-force term $\tau_\ell(u_i,\omega_j) - \tau_\ell(u_j,\omega_i)$ appears above, we could have used also the subscale force term $\epsilon_{ijk}f^s_{\ell k}$. Other forms of the relation may also be written. For example, again using the relations

$$
\partial_j\tau_\ell(\omega_j,u_i) = \tau_\ell(\omega_j,u_{i,j}) = \tau_\ell(\omega_j,S_{ij}),
$$

the small-scale enstrophy balance may be written as

$$
\begin{aligned}
\partial_t\zeta_\ell + \partial_k[\zeta_\ell\bar{u}_{\ell,k} &+ \frac{1}{2}\tau_\ell(\omega_i,\omega_i,u_k) - \nu\partial_k\zeta_\ell] \\
&= \tau_\ell(S_{ij},\omega_i,\omega_j) + \tau_\ell(\omega_i,\omega_j)\overline{S}_{\ell i,j} + \tau_\ell(\omega_i,S_{ij})\bar{\omega}_{\ell j} \\
&-\bar{\omega}_{\ell i,k}\tau_\ell(\omega_i,u_k) - \nu\tau_\ell(\omega_{i,k},\omega_{i,k}) \qquad (18)
\end{aligned}
$$

This is the version presented in T & L , eq.(3.3.38). Note that the production of enstrophy in the small-scales by flux from the large-scales

$$
-\bar{\omega}_{\ell i,j}(\tau_\ell(\omega_i,u_j) - \tau_\ell(\omega_j,u_i))
$$

[or the similar term that appears in the T & L versions] is exactly equal and opposite to the enstrophy flux out of the large-scales in (13). This is very similar to what occurred in the energy balances. However, we shall see that the enstrophy dynamics is NOT dominated by nonlinear cascade!