# IV Turbulent Vortex Dynamics

## (A) Energy Cascade and Vortex Dynamics

See T & L, Section 2.3, 8.2

We shall investigate more thoroughly the dynamical mechanism of forward energy cascade to small-scales, i.e. the origin of the irreversibility or time-asymmetry which leads to

$$\langle \Pi_\ell \rangle = -\langle \overline{\mathbf{S}}_\ell : \boldsymbol{\tau}_\ell \rangle \quad > \ 0.$$

It is useful here to begin by developing some illuminating approximate expressions for the turbulent stress $\boldsymbol{\tau}_\ell$. Our simplifications shall be based on the property of UV-locality that were previously established. This property states that

$$\boldsymbol{\tau}_\ell(\bar{\mathbf{u}}_\delta, \bar{\mathbf{u}}_\delta) \cong \boldsymbol{\tau}_\ell(\mathbf{u}, \mathbf{u}), \quad \delta \ll \ell.$$

Pushing this property to the limit, let's approximate $\boldsymbol{\tau}_\ell(\mathbf{u}, \mathbf{u})$ with the expression

$$\boldsymbol{\tau}_\ell^{\#}(\mathbf{u}, \mathbf{u}) \equiv \boldsymbol{\tau}_\ell(\bar{\mathbf{u}}_\ell, \bar{\mathbf{u}}_\ell) \quad \text{for} \quad \ell = \delta.$$

This approximation is essentially what is called the (modified) Leonard stress or the similarity model in the LES literature, see

> C. Meneveau & J. Katz, "Scale-invariance and turbulence models for large-eddy simulation," Annu. Rev. Fluid Mech. **32** 1-32 (2000)

As we have derived it here, the basic assumption is an extreme form of UV-locality. In addition to this approximation, we shall make one further simplification based on the fact that the filtered field $\bar{\mathbf{u}}_\ell$ is $C^\infty$ so that we may approximate the increments $\delta\bar{\mathbf{u}}_\ell(\mathbf{r})$ by the first term in their Taylor expansion:

$$\delta\bar{\mathbf{u}}_\ell(\mathbf{r}; \mathbf{x}) \cong (\mathbf{r} \cdot \boldsymbol{\nabla})\bar{\mathbf{u}}_\ell(\mathbf{x}).$$

If we then substitue into

$$\boldsymbol{\tau}_\ell(\bar{\mathbf{u}}_\ell, \bar{\mathbf{u}}_\ell) = \langle \delta\bar{\mathbf{u}}_\ell \delta\bar{\mathbf{u}}_\ell \rangle_\ell - \langle \delta\bar{\mathbf{u}}_\ell \rangle_\ell \, \langle \delta\bar{\mathbf{u}}_\ell \rangle_\ell$$

we get an expression

$$\boldsymbol{\tau}_\ell(\bar{\mathbf{u}}_\ell, \bar{\mathbf{u}}_\ell) \cong \ell^2 C_{ij} (\partial_i \bar{\mathbf{u}}_\ell)(\partial_j \bar{\mathbf{u}}_\ell)$$

with

$$
\begin{aligned}
C_{ij} &= \langle r_i r_j \rangle - \langle r_i \rangle \langle r_j \rangle \\
&= \int d^d r \; r_i r_j G(\mathbf{r}) - \int d^d r \; r_i G(\mathbf{r}) \int d^d r \; r_j G(\mathbf{r})
\end{aligned} \tag{1}
$$

The simplest formulation is obtained if one assumes that the filter kernel $G$ is spherically symmetric, so that

$$
\begin{aligned}
\langle r_i \rangle &= 0 \\
C_{ij} &= \langle r_i r_j \rangle = \frac{1}{d}\langle |r|^2 \rangle \delta_{ij} = \frac{1}{d} C_2 \cdot \delta_{ij}
\end{aligned} \tag{2}
$$

This yields the final expression

$$\boldsymbol{\tau}^*_\ell(\mathbf{u},\mathbf{u}) \equiv \tfrac{1}{d} C_2 \ell^2 \left(\tfrac{\partial \bar{\mathbf{u}}_\ell}{\partial \mathbf{x}}\right)\left(\tfrac{\partial \bar{\mathbf{u}}_\ell}{\partial \mathbf{x}}\right)^\top$$

with $C_2 = \langle |r|^2 \rangle = \int d^d r |\mathbf{r}|^2 G(\mathbf{r})$. This expression is also a model found in LES and called by various names, such as the Clark model, the tensor viscosity model or the nonlinear model. See again Meneveau & Katz(2000). The additional assumption behind this approximation is space-locality of the stress, which allows one to replace the space integral over increments with an expression in terms of local gradients $\boldsymbol{\nabla}\bar{\mathbf{u}}_\ell(\mathbf{x})$ evaluated at the point $\mathbf{x}$. It is known that the expression $\boldsymbol{\tau}^*_\ell(\mathbf{u},\mathbf{u})$ correlates very well with the true stress $\boldsymbol{\tau}_\ell(\mathbf{u},\mathbf{u})$, both in magnitude and in orientation. For example, see

> V. Borue & S. A. Orszag, "Local energy flux and subgrid-scale statistics in three-dimensional turbulence," J. Fluid Mech. **366** 1-31(1998)

These authors verify the model in a DNS study compared with the true stress, obtaining typical correlation coefficients of order 90% or higher. Thus, the approximations that we have made — though relatively crude — seem in fact to be adequate to explore the basic physics of the energy cascade process[1].

[1]NOTE: It can be shown that the approximations $\boldsymbol{\tau}^{\#}_\ell$ and $\boldsymbol{\tau}^*_\ell$ are just the first terms in a systematic expansion in scales and in gradients. See G. L. Eyink, "Multiscale gradient expansion of the turbulent stress tensor," J. Fluid Mech. **549** 159-190(2006)

To that end, we now introduce a standard decomposition $\frac{\partial \bar{\mathbf{u}}_\ell}{\partial \mathbf{x}} = \overline{\mathbf{S}}_\ell + \overline{\boldsymbol{\Omega}}_\ell$ using the symmetric $\overline{\mathbf{S}}_\ell$ and anti-symmetric $\overline{\boldsymbol{\Omega}}_\ell$ parts:

$$\overline{\mathbf{S}}_\ell = \tfrac{1}{2}[\left(\tfrac{\partial \bar{\mathbf{u}}_\ell}{\partial \mathbf{x}}\right) + \left(\tfrac{\partial \bar{\mathbf{u}}_\ell}{\partial \mathbf{x}}\right)^\top], \quad \overline{\boldsymbol{\Omega}}_\ell = \tfrac{1}{2}[\left(\tfrac{\partial \bar{\mathbf{u}}_\ell}{\partial \mathbf{x}}\right) - \left(\tfrac{\partial \bar{\mathbf{u}}_\ell}{\partial \mathbf{x}}\right)^\top].$$

This yields

$$\boldsymbol{\tau}^*_\ell(\mathbf{u},\mathbf{u}) = \tfrac{1}{d} C_2 \ell^2 \{\overline{\mathbf{S}}_\ell^2 - [\overline{\mathbf{S}}_\ell, \overline{\boldsymbol{\Omega}}_\ell] - \overline{\boldsymbol{\Omega}}_\ell^2\}.$$

Using the relation $\Omega_{ij} = -\frac{1}{2}\epsilon_{ijk}\omega_k$ in 3D, this can also be written as

$$\boldsymbol{\tau}^*_\ell = \frac{1}{3} C_2 \ell^2 \{\overline{\mathbf{S}}_\ell^2 + \frac{1}{2}(\bar{\boldsymbol{\omega}}_\ell \times \overline{\mathbf{S}}_\ell - \overline{\mathbf{S}}_\ell \times \bar{\boldsymbol{\omega}}_\ell) + \frac{1}{4}(\mathbf{I}|\bar{\boldsymbol{\omega}}_\ell|^2 - \bar{\boldsymbol{\omega}}_\ell \bar{\boldsymbol{\omega}}_\ell)\} \quad \text{(for } d = 3\text{)} \qquad (3)$$

The last term has a very nice physical interpretation: it is a tensile (positive) stress in the plane orthogonal to vorticity vector $\bar{\boldsymbol{\omega}}_\ell$. This stress contribution can be understood from simple vortex models of the subscale modes, e.g. see A. Misra and D. I. Pullin, "A vortex-based subgrid stress model for large-eddy simulation," Phys. Fluids **9** 2443-2454(1997). Intuitively, a more-or-less rectilinear vortex tube has the largest velocity components in the plane orthogonal to its aixs:

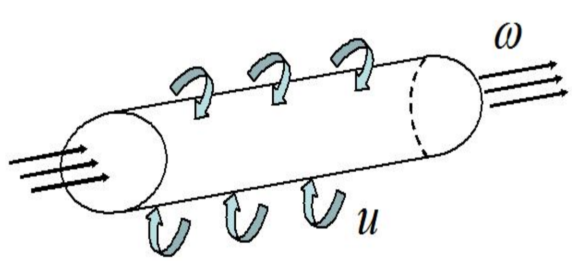


Figure 1. a rectilinear vortex tube

Analytical models of this type can explain also the other terms in Eq.(3). For example, consider a circular vortex tube of strength $\omega_0$ with rectilinear axis aligned along the $z$-direction. In cylindrical coordinates, the velocity becomes

$$\mathbf{u}(\mathbf{x}) = \begin{cases} \frac{1}{2}\omega_0 \hat{\mathbf{z}} \times \mathbf{x} & r \leq R \\ \frac{1}{2}\omega_0 (\frac{R}{r})^2 \hat{\mathbf{z}} \times \mathbf{x} & r > R \end{cases}$$

where $R$ is the radius of the tube, the vorticity of the tube is

$$\boldsymbol{\omega}(\mathbf{x}) = \begin{cases} \omega_0 \hat{\mathbf{z}} & r \leq R \\ 0 & r > R \end{cases}$$

while the strain is

$$\mathbf{S}(\mathbf{x}) = \frac{1}{2}\omega_0 (\tfrac{R}{r})^2 \begin{bmatrix} \sin(2\theta) & -\cos(2\theta) & 0 \\ -\cos(2\theta) & \sin(2\theta) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ r > R.$$

and $= \mathbf{0}$ for $r \leq R$. Thus, the vorticity and the strain are in complementary distribution, with $\boldsymbol{\omega}$ non-zero inside the tube and $\mathbf{S}$ non-zero outside. Neverthess, the velocity itself is non-zero everywhere and lies entirely in the plane orthogonal to the axis. Thus, one should get non-vanishing stress contributions from both inside & outside.

Substituting $\boldsymbol{\nabla}\mathbf{u}(\mathbf{x})$ for the vortex tube (with $R = \ell$) into the formula (3) yields

$$\boldsymbol{\tau}_\ell^*(\mathbf{x}) = \frac{1}{12} C_2 (\omega_0 \ell)^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times \begin{cases} 1 & r \leq \ell \\ (\frac{\ell}{r})^4 & r > \ell \end{cases}$$

The cross term $[\overline{\mathbf{S}}_\ell, \overline{\boldsymbol{\Omega}}_\ell]$ can also be obtained in vortex tube models with flattened cores, such as the Neu model of a rectilinear vortex tube with elliptical cross-sections (J. C. Neu, Phys. Fluids **27** (10) 2397(1984)). In such flattened tubes, there is strain as well as vorticity inside the volume of the tube.

An important conclusion from these considerations is that vortex tubes tend to induce a dominant tensile stress in the plane orthogonal to their axis. Equivalently, this corresponds to a contractile (negative) stress along the vortex axis, by incompressibility. Note that only the deviatoric part $\mathring{\boldsymbol{\tau}}_\ell$ contributes non-trivially to the subscale force $\mathbf{f}_\ell^s = -\boldsymbol{\nabla} \cdot \mathring{\boldsymbol{\tau}}_\ell$, while the diagonal term $-\frac{1}{d}\boldsymbol{\nabla} k_\ell$ can be absorbed into a redefinition of the pressure. Thus, we see that vortex tubes tend to induce a turbulent stress which is like that of an "elastic band", so that the tubes would naturally tend to decrease in length due to the flow induced by their own stress. This means that, also, stretching of vortex tubes by large-scale strain requires an expenditure of energy by the strain to overcome this opposing stress.

This may be seen clearly in the expression for the energy flux that follows from $\boldsymbol{\tau}_\ell^*$:

$$\begin{aligned} \Pi_\ell^* &= -\overline{\mathbf{S}}_\ell : \boldsymbol{\tau}_\ell^* \\ &= \frac{1}{3} C_2 \ell^2 [-\mathrm{tr}(\overline{\mathbf{S}}_\ell^3) + \frac{1}{4} \bar{\boldsymbol{\omega}}_\ell^\top \overline{\mathbf{S}}_\ell \bar{\boldsymbol{\omega}}_\ell] \end{aligned} \tag{4}$$

Note that

$$\mathrm{tr}(\overline{\mathbf{S}}_\ell[\overline{\mathbf{S}}_\ell, \overline{\boldsymbol{\Omega}}_\ell]) = 0$$

by cyclicity of the trace. The above formula seems to have been first observed by Borue & Orszag(1998), who also showed that it is a very accurate approximation to the true $\Pi_\ell$. The contribution proportional to $\bar{\boldsymbol{\omega}}_\ell^\top \overline{\mathbf{S}}_\ell \bar{\boldsymbol{\omega}}_\ell$ — or, the stretching rate of large-scale vorticity — is positive when the lines of the large-scale vorticity are aligned with the positive (stretching) direction of the large-scale strain. In fact,

$$\langle \Pi_\ell^* \rangle = \tfrac{1}{3} C_2 \ell^2 \langle \bar{\boldsymbol{\omega}}_\ell^\top \overline{\mathbf{S}}_\ell \bar{\boldsymbol{\omega}}_\ell \rangle$$

by appealing to a famous result of R. Betchov, "An inequality concerning the production of vorticity in isotropic," J. Fluid Mech. **1** 497-504(1956):

$$-\langle \mathrm{tr}(\overline{\mathbf{S}}_\ell^3) \rangle = \tfrac{3}{4} \langle \bar{\boldsymbol{\omega}}_\ell \overline{\mathbf{S}}_\ell \bar{\boldsymbol{\omega}}_\ell \rangle$$

for any homogeneous average. This is usually called Betchov's relation. This relation is purely kinematic not dynamic, i.e. it is based only on homogeneity and incompressibility and does not make any use of the Navier-Stokes equation!

Proof: The identity

$$2u_{i,j}u_{j,k}u_{k,i} = (u_i u_{j,k}u_{k,i})_j - (u_i u_{j,k}u_{k,j})_i + (u_i u_{j,i}u_{k,j})_k$$

is easily checked, using incompressibility of $\mathbf{u}$. Thus,

$$\langle \mathrm{tr}[(\boldsymbol{\nabla}\mathbf{u})^3]\rangle = 0$$

for any homogenous average or for a space-average over any domain with $\mathbf{u} = 0$ at the boundary. Substituting $\boldsymbol{\nabla}\mathbf{u} = \mathbf{S} + \boldsymbol{\Omega}$,

$$0 = \langle \mathrm{tr}(\mathbf{S}^3)\rangle + 3\underbrace{\langle \mathrm{tr}(\mathbf{S}^2\boldsymbol{\Omega})\rangle}_{=0} + 3\langle \mathrm{tr}(\mathbf{S}\boldsymbol{\Omega}^2)\rangle + \underbrace{\langle \mathrm{tr}(\boldsymbol{\Omega}^3)\rangle}_{=0} \quad (5)$$

where the terms set $= 0$ vanish due to $\mathbf{S}^\top = \mathbf{S}, \boldsymbol{\Omega}^\top = -\boldsymbol{\Omega}$ and the cyclicity and transpose-invariance of the trace. Using $\Omega_{ij} = -\frac{1}{2}\epsilon_{ijk}\omega_k$, we finally get the stated relation. QED!

Remarks:

(i) The quantity $-\langle \mathrm{tr}(\mathbf{S}^3)\rangle$ is called the negative strain skewness. Betchov's relation implies that 75% of the mean flux $\langle \Pi^*_\ell\rangle$ comes from negative strain skewness and 25% from the direct vortex-stretching term.

(ii) Another identity of the same type follows from

$$u_{i,j}u_{j,i} = \partial_i\partial_j(u_i u_j) = \partial_j(u_i u_{j,i})$$

so that

$$\langle \mathrm{tr}[(\boldsymbol{\nabla}\mathbf{u})^2]\rangle = 0$$

again for any homogenous average. This gives

$$0 = \langle \mathrm{tr}(\mathbf{S}^2)\rangle + 2\underbrace{\langle \mathrm{tr}(\mathbf{S}\boldsymbol{\Omega})\rangle}_{=0} + \langle \mathrm{tr}(\boldsymbol{\Omega}^2)\rangle \quad (6)$$

or, with $\mathrm{tr}(\boldsymbol{\Omega}^2) = -\frac{1}{2}|\boldsymbol{\omega}|^2$,

$$\langle |\mathbf{S}|^2\rangle = \tfrac{1}{2}\langle |\boldsymbol{\omega}|^2\rangle.$$

The most important conclusion is that, insofar as $\Pi^*_\ell$ is a valid model for energy flux,

$$\langle \Pi^*_\ell\rangle = \tfrac{1}{3}C_2\ell^2\langle \bar{\boldsymbol{\omega}}_\ell^\top \overline{\mathbf{S}}_\ell \bar{\boldsymbol{\omega}}_\ell\rangle$$

and mean energy flux to small-scales is directly propotional to the mean rate of vortex-stretching. This does not yet answer the main question: how and why does it happen that

$$\langle \Pi_\ell \rangle \cong \langle \Pi_\ell^* \rangle \ > 0 \ \ ???$$

The question is merely shifted: why is it the case that $\langle \bar{\boldsymbol{\omega}}_\ell^\top \overline{\mathbf{S}}_\ell \bar{\boldsymbol{\omega}}_\ell \rangle \ > 0$ ??? The rest of this chapter is devoted to understanding this issue, insofar as is possible. We cannot be definitive, because the problem is still very open.

We shall focus mainly on a view which originated, essentially, with G. I. Taylor in 1937/1938. This idea attempts to explain

$$\langle \bar{\boldsymbol{\omega}}_\ell^\top \overline{\mathbf{S}}_\ell \bar{\boldsymbol{\omega}}_\ell \rangle \ > 0$$

by the hypothesis that vortex-lines will, on average, tend to lengthen in a turbulent flow. Taylor's idea was originally proposed to explain the enhancement of viscous dissipation $\nu\langle|\boldsymbol{\omega}|^2\rangle$ and there was applied to fine-grained vorticity. However, his ideas can be carried over to coarse-grained vorticity $\bar{\boldsymbol{\omega}}_\ell$ in the inertial-range. Consider his following statement:

> "The author has put forward the view that the high average vorticity which is known to exist in turbulent motion is caused by extension of vortex filaments in an eddying fluid. Let $A$ and $B$ be two particles a short distance, $d_0$, apart on a vortex line where the vorticity is $\omega_0$. At a subsequent time when the distance between $A$ and $B$ has changed from $d_0$ to $d$ and the vorticity from $\omega_0$ to $\omega$ then, neglecting the effects of viscosity, the equation representing the conservation of circulation is
>
> $$\omega/\omega_0 = d/d_0. \qquad (1)$$
>
> Turbulent motion is found to be diffusive, so that particles which were originally neighbors move apart as the motion proceeds. In a diffusive motion the average value of $d^2/d_0^2$ continually increases. It will be seen therefore from (1) that the average value of $\omega^2/\omega_0^2$ continually increases."
>
> — G. I. Taylor, "Production and dissipation of vorticity in a turbulent fluid," Proc. Roy. Soc. Lond., Sec. A. **164**, 15-23(1938)

There is nothing in the above ideas that restricts their applicability to the fine-grained vorticity, in the dissipation range. In fact, Taylor's remark about "neglecting the effects of viscosity" is much more justifiable for coarse-grained vorticity in the inertial-range, as we have seen.

The first explicit attempt to apply Taylor's ideas to inertial-range dynamics seems to have been by L. Onsager. For example, in 1945 he wrote the following in a letter to C. C. Lin proposing the K41 theory (independently of Kolmogorov):

> "In terms of the Lagrangian description, the dissipation of energy in turbulent motion must be attributed to stretching of the vortex fibres, which generates vorticity more rapidly the more vigorous the motion and thus accelerates the final dissipation by viscosity (Taylor).
>
> ...
>
> The distribution law (19) [the $k^{-5/3}$ energy spectrum] is compatible with the hypothesis that the mean rate of stretching of vortex lines is given by the average rate of deformation in the liquid."
>
> — L. Onsager, letter to C. C. Lin, June 1945, Papers of Theodore von *Kármán*, California Institue of Technology, Box 18, Folder 22.

Here Onsager seems to have in mind that the average rate of stretching of vortex lines of $\bar{\boldsymbol{\omega}}_\ell$ at length-scale $\ell$ is set by the local turnover time

$$\tau_\ell \sim 1/|\boldsymbol{\nabla}\bar{\mathbf{u}}_\ell| \sim \epsilon^{-1/3}\ell^{2/3}$$

which is determined by the rate of deformation $\boldsymbol{\nabla}\bar{\mathbf{u}}_\ell = \delta u(\ell)/\ell$ at that scale and the K41 scaling $\delta u(\ell) \sim (\epsilon\ell)^{1/3}$. Onsager also discussed Taylor's ideas again in his own 1949 paper:

"However, as pointed out by G. I. Taylor [7], convection in three dimensions will tend to increase the total vorticity. Since the circulation of a vortex tube is conserved, the vorticity will increase whenever a vortex tube is stretched. Now it is very reasonable to expect that a vortex line — or any line which is deformed by the motion of the liquid — will tend to increase in length as a result of more or less haphazard motion. This process tends to make the texture of the motion ever finer, and greatly accelerates the viscous dissipation."

— L. Onsager, "Statistical hydrodynamics," Nuovo Cimento Suppl. **6** 279-287(1949)

One idea that is clearly expressed here is the notion that the irreversibility of the turbulent dissipation process is connected with the statistical tendency of vortex-lines (or indeed, of any material lines) to lengthen under random turbulent advection.

This same idea is also repeated later by Feynman, with some additional twists:

"In ordinary fluids flowing rapidly and with very low viscosity the phenomenon of turbulent sets in. A motion involving vorticity is unstable. The vortex lines twist about in an ever more complex fashion, increasing their length at the expense of the kinetic energy of the main stream. That is, if a liquid is flowing at a uniform velocity and a vortex line is started somewhere upstream, this line is twisted into a long complex tangle further downstream. To the uniform velocity is added a complex, irregular velocity field. The energy for this is supplied by pressure head."

— R. P. Feynman, "Application of Quantum Mechanics to Liquid Helium," In Progress in Low Temperature Physics, vol.1, ed. by C. J. Gorter (North-Holland Publishing Co., Amsterdam, 1955), Chapter II, p.17-53

The idea that vortex-stretching is the basic process of 3D turbulent energy cascade is still probably the majority view today. For example, consider the following statements from the textbook of T &L:

"The existence of energy transfer from large eddies to small eddies, driven by vortex-stretching and leading to viscous dissipation of energy near the Kolmogorov microscale, was demonstrated in Chapter 3. Here, we discuss how the energy exchange takes place. Let us briefly recall the vortex-stretching mechanism. When vorticity finds itself in a strain-rate field, it is subject to stretching. On the basis of conservation of angular momentum, we expect that the vorticity in the direction of a positive strain rate is amplified, while the vorticity in the direction of a negative strain rate is attenuated.

...

Vortex stretching involves an exchange of energy, because the strain rate performs deformation work on the vortices that are being stretched."

- H. Tennekes & J. L. Lumley, A First Course in Turbulence, (MIT Press, Cambridge, MA 1972), Section 8.2, p. 256-7

We see that all of these illustrious authors - as well as many others - agree that vortex-stretching is the basic process of turbulent energy dissipation!

However, despite the superficial agreement, there are also clear differences in the ideas expressed. The above quotes are actually discussing several, quite different processes! Taylor(1938) and Onsager(1949) discuss the role of vortex stretching in viscous energy dissipation. Instead, Onsager(1945) and Tennekes & Lumley(1972) consider how vortex-stretching is involved in inertial-range energy cascade. Finally, Feynman(1955) is concerned with a still different process, the contribution of vortex-stretching to pressure-drop down a pipe. It is not clear precisely how all of these distinct vortex-stretching mechanisms are related!

It is the purpose of this chapter to study in some detail and with a critical eye, all of these proposed ideas on vortex stretching. We shall find that there is considerable support for the proposed vortex-stretching mechanism, and, indeed, we find the case compelling that vortex-stretching is the main engine that drives three-dimensional turbulence. However, the problem has great complexity and subtlety. For example, we shall see that NONE of the ideas proposed above is even internally self-consistent! All of the above proposals are based on assumptions that are clearly untrue, at least in a direct and most obvious sense. For example, the above theoretical pictures assume that the Kelvin Theorem on conservation of circulations holds in a naive sense. (This is stated explicitly by Taylor & Onsager and it is what Tennekes & Lumley really mean by "conservation of angular momentum.") However, we shall see that conservation of circulations cannot hold — at least in any naive sense — in either the dissipation or the inertial range of turbulent flow. Also, Feynman talks about a vortex line being "twisted into a long complex tangle" that leads to a "complex, irregular velocity field." However, we shall see that pressure drop down a pipe requires organized motion of vorticity, not a purely stochastic, random stretching.

Like all of the above authors, however, we believe that vortex-stretching is a key to understanding 3D turbulence. Therefore, it is worth a very careful investigation by theory, numerics, and experiment!