(E) Locality of Energy Transfer

See T & L, Section 8.2; U. Frisch, Section 7.3

<u>The Essence of the Matter</u>

We have seen that energy is transferred from scales $> \ell$ to scales $< \ell$ by the <u>deformation work</u>

$$\Pi_\ell(\mathbf{x}) = -\bar{\mathbf{S}}_\ell(\mathbf{x}) : \boldsymbol{\tau}_\ell(\mathbf{x})$$

As we now discuss, this transfer is <u>scale-local</u> under conditions that are realistic for turbulent flow. Suppose that the velocity $\mathbf{v}$ is <u>Hölder continuous</u> at point $\mathbf{x}$ with exponent $0 < h < 1$, i.e.

$$\delta\mathbf{v}(\mathbf{r};\mathbf{x}) = O(|\mathbf{r}|^h).$$

For example, in K41 theory, $h = 1/3$ at every point $\mathbf{x}$ in the flow. Then, as follows from our earlier discussion,

$$\bar{\mathbf{S}}_\ell(\mathbf{x}) = O(\ell^{h-1})^\star$$

and

$$\boldsymbol{\tau}_\ell(\mathbf{x}) = O(\ell^{2h})^{\star\star}$$

so that

$$\Pi_\ell = O(\ell^{3h-1})$$

at the point $\mathbf{x}$. We have proved these only as upper bounds, but let us assume, for the sake of argument, that $= O(\ell^\alpha)$ here means in fact $\sim (const.)\ell^\alpha$. (We'll return to this issue later!)

Where does most of the strain come from? We can consider a larger length-scale Δ, with $\ell \ll \Delta \ll L$ and write

$$\bar{\mathbf{S}}_\ell(\mathbf{x}) = \bar{\mathbf{S}}_\Delta(\mathbf{x}) + \mathbf{S}_{[\ell,\Delta]}(\mathbf{x})$$

where

$$\begin{aligned} \bar{\mathbf{S}}_\Delta &= \text{strain from scales } > \Delta \\ \mathbf{S}_{[\ell,\Delta]} &= \text{band-pass filtered strain from scales between } \ell \text{ and } \Delta \end{aligned}$$

But the previous estimates apply to $\bar{\mathbf{S}}_\Delta$, so that

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$\star$ requires $\int d^d\rho\, |\boldsymbol{\rho}|^h |\boldsymbol{\nabla} G(\boldsymbol{\rho})| < +\infty$. $\star\star$ requires $\int d^d\rho\, |\boldsymbol{\rho}|^{2h} G(\boldsymbol{\rho}) < +\infty$

$$\bar{\mathbf{S}}_\Delta(\mathbf{x}) = O(\Delta^{h-1}).$$

In that case,

$$\frac{|\bar{\mathbf{S}}_\Delta(\mathbf{x})|}{|\bar{\mathbf{S}}_\ell(\mathbf{x})|} = O(\frac{\Delta^{h-1}}{\ell^{h-1}}) = O((\frac{\ell}{\Delta})^{1-h})$$

which is $\ll 1$ whenever $h < 1$. Thus, we see that most of the strain $\bar{\mathbf{S}}_\ell(\mathbf{x})$ comes from scales near ℓ and very little comes from scales $\Delta \gg \ell$, whenever $h < 1$.

Now, what about the stress from small-scales? We can likewise consider a smaller length δ, with $\eta \ll \delta \ll \ell$ and write

$$\boldsymbol{\tau}_\ell(\mathbf{x}) = \boldsymbol{\tau}_\delta(\mathbf{x}) + \boldsymbol{\tau}_{[\delta,\ell]}(\mathbf{x})$$

where

$$\begin{aligned} \boldsymbol{\tau}_\delta(\mathbf{x}) &= \boldsymbol{\tau}_\delta(\mathbf{v},\mathbf{v}) = \text{ stress from scales } < \delta \\ \boldsymbol{\tau}_{[\delta,\ell]}(\mathbf{x}) &= \text{ stress from scales between } \delta \text{ and } \ell \end{aligned}$$

The above equation defines $\boldsymbol{\tau}_{[\delta,\ell]}$. (NOTE: A better way to do this is by using the so-called Germano identity. This will be explored in the homework!) But, again, the previous estimates apply to $\boldsymbol{\tau}_\delta$, so that

$$\boldsymbol{\tau}_\delta(\mathbf{x}) = O(\delta^{2h}).$$

In that case

$$\frac{|\boldsymbol{\tau}_\delta(\mathbf{x})|}{|\boldsymbol{\tau}_\ell(\mathbf{x})|} = O(\frac{\delta^{2h}}{\ell^{2h}}) = O((\frac{\delta}{\ell})^{2h})$$

which is $\ll 1$ whenever $h > 0$. Thus, we see that most of the stress $\boldsymbol{\tau}_\ell(\mathbf{x})$ comes from scales near ℓ and very little comes from scales $\delta \ll \ell$, whenever $h > 0$.

The conclusion is that the energy transfer from length-scales $> \ell$ to length-scales $< \ell$ is dominated by interactions of modes at scales $\approx \ell$. In fact, two modes with length scales $> \ell$ can, by quadratic nonlinearities, interact only with modes down to length scales $> \ell/2$. This is a basic result of Fourier analysis, since, if two modes have only wavenumbers $\mathbf{k}, \mathbf{k}'$ such that

$$|\mathbf{k}|, |\mathbf{k}'| < \frac{2\pi}{\ell}$$

then their product can contain wavenumbers $\mathbf{k}''$ with at most

$$
\begin{aligned}
|\mathbf{k}''| = |\mathbf{k}+\mathbf{k}'| &\leq |\mathbf{k}| + |\mathbf{k}'| \\
&\leq \frac{2\pi}{\ell} + \frac{2\pi}{\ell} = \frac{4\pi}{\ell} = \frac{2\pi}{\ell/2}
\end{aligned}
$$

The picture that emerges is of energy transfer across the length-scale ℓ by interaction of modes with scale near ℓ, to a length-scale $\gtrsim \frac{\ell}{2}$. The energy at this scale is then, in turn, transferred by similar scale-local interactions to a length-scale $\gtrsim \frac{\ell}{4}$, and so on. This stepwise process is called a local cascade (in this case, of energy). Here it is crucial that only modes with scale $\approx \ell$ participate in the transfer of excitation across the length-scale ℓ.

If the individual steps in the cascade process are also chaotic nonlinear process, then it is reasonable to expect that the small-scales will "forget" about the detailed geometry and statistics of the large-scale flow modes. In particular, there is no "direct communication" with large-scale modes in the process which creates and maintains the small-scale motions.

These considerations motivate the idea of universality of the small-scales, that is, the notion that the statistics of the small-scale modes shall be the same for all flows and independent of the details of the large-scale geometry, generation mechanisms, etc. In particular, the symmetries of the dynamics — space homogeneity, temporal invariance, rotational isotropy, scale invariance,etc. — should be restored on a statistical level.

This university — and thus also scale-locality — is quite important for the physical foundation of large-eddy simulation (LES) modelling of turbulent flow. It raises the hope that generally applicable (universal) models of small-scale stress $\boldsymbol{\tau}_\ell$ may be possible!

A More Precise Statement of Scale Locality

The filtered velocity gradient

$$\bar{\mathbf{D}}_\ell(\mathbf{v}) = \boldsymbol{\nabla}\bar{\mathbf{v}}_\ell$$

is a linear functional $\mathbf{D}_\ell$ of the velocity field $\mathbf{v}$. Likewise,

$$\boldsymbol{\tau}_\ell(\mathbf{v},\mathbf{v}) = \overline{(\mathbf{v}\mathbf{v})}_\ell - \bar{\mathbf{v}}_\ell\bar{\mathbf{v}}_\ell$$

is a quadratic functional of $\mathbf{v}$. The concept of infrared (IR) or large-scale locality is that replacing any $\mathbf{v}$ with $\bar{\mathbf{v}}_\Delta$ will lead to a much smaller contribution, for $\Delta \gg \ell$. Conversely, this means that most of the contribution will be obtained by replacing $\mathbf{v}$ with $\mathbf{v}'_\Delta$, for a sufficiently large $\Delta \gg \ell$. The concept of ultraviolet (UV) or small-scale locality is that replacing any $\mathbf{v}$ with $\mathbf{v}'_\delta$ will lead to a much smaller contribution, for $\delta \ll \ell$. Again, this means that most of the contribution will be obtained by replacing $\mathbf{v}$ with $\bar{\mathbf{v}}_\delta$, for a sufficiently small $\delta \ll \ell$.

These results hold if the velocity field $\mathbf{v}$ is Hölder continuous at space point $\mathbf{x}$ with exponent $0 < h < 1$, as the following precise estimates show:

IR Locality: For $\Delta > \ell$,

$$|\bar{\mathbf{D}}_\ell(\bar{\mathbf{v}}_\Delta| = |\bar{\mathbf{D}}_\ell(\mathbf{v})| \cdot O((\tfrac{\ell}{\Delta})^{1-h}), \quad \text{or} \quad \bar{\mathbf{D}}_\ell(\mathbf{v}'_\Delta) = \bar{\mathbf{D}}_\ell(\mathbf{v}) \cdot [1 + O((\tfrac{\ell}{\Delta})^{1-h})]$$

$$|\boldsymbol{\tau}_\ell(\bar{\mathbf{v}}_\Delta,\mathbf{v})| = |\boldsymbol{\tau}_\ell(\mathbf{v},\mathbf{v})| \cdot O((\tfrac{\ell}{\Delta})^{1-h}), \quad \text{or} \quad \boldsymbol{\tau}_\ell(\mathbf{v}'_\Delta,\mathbf{v}) = \boldsymbol{\tau}_\ell(\mathbf{v},\mathbf{v}) \cdot [1 + O((\tfrac{\ell}{\Delta})^{1-h})].$$

Note, BTW, that replacing both $\mathbf{v}'$s with $\bar{\mathbf{v}}'_\ell$s in $\boldsymbol{\tau}_\ell$ leads to an even smaller result:

$$|\boldsymbol{\tau}_\ell(\bar{\mathbf{v}}_\Delta,\bar{\mathbf{v}}_\Delta)| = |\boldsymbol{\tau}_\ell(\mathbf{v},\mathbf{v})| \cdot O((\tfrac{\ell}{\Delta})^{2(1-h)})$$

UV Locality: For $\delta < \ell$,

$$|\bar{\mathbf{D}}_\ell(\mathbf{v}'_\delta)| = |\bar{\mathbf{D}}_\ell(\mathbf{v})| \cdot O((\tfrac{\delta}{\ell})^{h}), \quad \text{or} \quad \bar{\mathbf{D}}_\ell(\bar{\mathbf{v}}_\delta) = \bar{\mathbf{D}}_\ell(\mathbf{v}) \cdot [1 + O((\tfrac{\delta}{\ell})^{h})]$$

$$|\boldsymbol{\tau}_\ell(\mathbf{v}'_\delta,\mathbf{v})| = |\boldsymbol{\tau}_\ell(\mathbf{v},\mathbf{v})| \cdot O((\tfrac{\delta}{\ell})^{h}), \quad \text{or} \quad \boldsymbol{\tau}_\ell(\bar{\mathbf{v}}_\delta,\mathbf{v}) = \boldsymbol{\tau}_\ell(\mathbf{v},\mathbf{v}) \cdot [1 + O((\tfrac{\delta}{\ell})^{h})].$$

and, of course,

$$|\boldsymbol{\tau}_\ell(\mathbf{v}'_\delta,\mathbf{v}'_\delta)| = |\boldsymbol{\tau}_\ell(\mathbf{v},\mathbf{v})| \cdot O((\frac{\delta}{\ell})^{2h}).$$

For a complete discussion and proof, see

G. L. Eyink, “Locality of turbulent cascades,” Physica D, **207**, 91-116 (2005)

However, the basic point is that $\bar{\mathbf{D}}_\ell(\mathbf{v})$, $\boldsymbol{\tau}_\ell(\mathbf{v},\mathbf{v})$ are — as we have seen long ago — given by integrals of velocity increments $\delta\mathbf{v}(\mathbf{r})$ for $|\mathbf{r}|\leq \ell$ (essentially). Furthermore, it is a consequence of the following lemma that the velocity increments themselves are scale-local:

Lemma: If $\mathbf{v}$ is Hölder continuous at point $\mathbf{x}$ with exponent $0<h<1$, then for $\Delta\geq\ell$

(1) $\delta\bar{\mathbf{v}}_\Delta(\boldsymbol{\ell};\mathbf{x})=O(\ell\Delta^{h-1})$, (2) $\delta\mathbf{v}'_\Delta(\boldsymbol{\ell};\mathbf{x})=\delta\mathbf{v}(\boldsymbol{\ell};\mathbf{x})+O(\ell\Delta^{h-1})=\delta\mathbf{v}(\boldsymbol{\ell};\mathbf{x})[1+O((\frac{\ell}{\Delta})^{1-h})]$

and for $\delta\leq\ell$

(3) $\delta\mathbf{v}'_\delta(\boldsymbol{\ell};\mathbf{x})=O(\delta^h)$, (4) $\delta\bar{\mathbf{v}}_\delta(\boldsymbol{\ell};\mathbf{x})=\delta\mathbf{v}(\boldsymbol{\ell};\mathbf{x})+O(\delta^h)=\delta\mathbf{v}(\boldsymbol{\ell};\mathbf{x})[1+O((\frac{\delta}{\ell})^h)]$,

if $\mathbf{v}$ is Hölder continuous with exponent $0<h<1$ in a neighbourhood of the point $\mathbf{x}$.

Proof: Note that (1)&(2) are, in fact, equivalent, since $\delta\mathbf{v}'_\Delta(\boldsymbol{\ell};\mathbf{x})=\delta\mathbf{v}(\boldsymbol{\ell};\mathbf{x})-\delta\bar{\mathbf{v}}_\Delta(\boldsymbol{\ell};\mathbf{x})$. Likewise, (3)&(4) are also equivalent.

The proof of (3) is simple, because

$$\delta\mathbf{v}'_\delta(\boldsymbol{\ell};\mathbf{x})=\mathbf{v}'_\delta(\mathbf{x}+\boldsymbol{\ell})-\mathbf{v}'_\delta(\mathbf{x}).$$

However, if $\boldsymbol{\ell}$ is close enough to zero, then both $\mathbf{x}+\boldsymbol{\ell}$ and $\mathbf{x}$ are in the neighbourhood with exponent h and thus

$$\mathbf{v}'_\delta(\mathbf{x})=-\int d\mathbf{r}\,G_\delta(\mathbf{r})\delta\mathbf{v}(\mathbf{r};\mathbf{x})=O(\delta^h)$$

and likewise for $\mathbf{v}'_\delta(\mathbf{x}+\boldsymbol{\ell})$.

The proof of (1) is just a bit more complex:

$$\begin{aligned}
\delta\bar{\mathbf{v}}_\Delta(\boldsymbol{\ell};\mathbf{x}) &= \int d\mathbf{r}\; G_\Delta(\mathbf{r})\quad \underbrace{\delta\mathbf{v}(\boldsymbol{\ell};\mathbf{x}+\mathbf{r})}_{[\mathbf{v}(\mathbf{x}+\mathbf{r}+\boldsymbol{\ell})-\mathbf{v}(\mathbf{x}+\mathbf{r})]}\\
&= \int d\mathbf{r}[G_\Delta(\mathbf{r}-\boldsymbol{\ell})-G_\Delta(\mathbf{r})]\mathbf{v}(\mathbf{x}+\mathbf{r})\\
&= \int d\mathbf{r}[G_\Delta(\mathbf{r}-\boldsymbol{\ell})-G_\Delta(\mathbf{r})]\,\underbrace{[\mathbf{v}(\mathbf{x}+\mathbf{r})-\mathbf{v}(\mathbf{x})]}_{\delta\mathbf{v}(\mathbf{r};\mathbf{x})}\\
&= -\frac{1}{\Delta}\int_0^1 d\theta\int d\mathbf{r}\;\boldsymbol{\ell}\cdot(\boldsymbol{\nabla}G)_\Delta(\mathbf{r}-\theta\boldsymbol{\ell})\delta\mathbf{v}(\mathbf{r};\mathbf{x})\\
&= O(\ell\Delta^{h-1})\qquad \underline{\text{QED!}}
\end{aligned}$$

Remark: The idea behind the last estimate is very simple: it is just a consequence of the fact that $\bar{\mathbf{v}}_\Delta$ is smooth. Thus,

$$\delta\bar{\mathbf{v}}_\Delta(\boldsymbol{\ell})\cong\boldsymbol{\ell}\cdot\boldsymbol{\nabla}\bar{\mathbf{v}}_\Delta=O(\ell\Delta^{h-1})$$

The first estimate is even simpler: $\delta\mathbf{v}'_\delta(\boldsymbol{\ell})$ is small, because the small-scale fluctuation field $\mathbf{v}'_\delta = O(\delta^h)$ is small! This is the essence of scale-locality. Note that these facts about velocity-increments were stated by Kolmogorov in the first of his 1941 papers on turbulence and also by Onsager in his 1945 letter to Lin and von Kármán.

Some important comments:

$\star$ There are some shortcomings in the above locality results, which we shall discuss later in the chapter on intermittency & scaling. In particular, it is not reasonable to assume pointwise scaling $\delta\mathbf{v}(\boldsymbol{\ell};\mathbf{x}) \sim (const)\ell^h$ and thus some of the estimates above are based on unrealistic assumptions. E.g. in the lemma above, (2a) & (4a) are OK, but (2b) & (4b) are not. For most purposes, (2a),(4a) suffice.

$\star$ We have been deriving upper bounds on nonlocal contributions. However, we have used only very simple estimates and the true contributions could be considerably smaller. For example, our estimate on the small-scale stress contribution to energy flux is

$$\begin{aligned} \Pi_\ell^{\text{stress}<\delta} &= -\bar{\mathbf{S}}_\ell : \boldsymbol{\tau}_\ell(\mathbf{v}'_\delta, \mathbf{v}'_\delta) \\ &= \Pi_\ell \cdot O((\frac{\delta}{\ell})^{2h}) \end{aligned}$$

Thus, the relative contribution of $\Pi_\ell^{\text{stress}<\delta}$ is at most $O((\frac{\delta}{\ell})^{2/3})$ for the K41 value $h = \frac{1}{3}$. However, this argument assumes that $\bar{\mathbf{S}}_\ell$ and $\boldsymbol{\tau}_\ell^{<\delta} = \boldsymbol{\tau}_\ell(\mathbf{v}'_\delta, \mathbf{v}'_\delta)$ are well-correlated, when, in fact, they are not. Notice that $\bar{\mathbf{S}}_\ell$ is associated to scales $> \ell$, while $\boldsymbol{\tau}_\ell^{<\delta}$ is associated to scales $< \delta$. In integrals over space (or over ensemble), there shall be large cancellations that are not considered in our rather crude estimates. Thus, the averages will be smaller by a factor of the correlation coefficient $\rho(\bar{\mathbf{S}}_\ell, \boldsymbol{\tau}_\ell^{<\delta})$, which can be estimated as in T&L, Section 8.2, p.261:

$$\rho(\bar{\mathbf{S}}_\ell, \boldsymbol{\tau}_\ell^{<\delta}) \sim \frac{t_\delta}{t_\ell} = O((\frac{\delta}{\ell})^{1-h})$$

where $t_\ell = \ell/\delta v(\ell)$ is the turnover time at length ℓ. Thus, in K41 theory one can argue that

$$\langle \Pi_\ell^{\text{stress}<\delta} \rangle = \langle \Pi_\ell \rangle \cdot O((\frac{\delta}{\ell})^{2/3} \cdot (\frac{\delta}{\ell})^{2/3}) = \langle \Pi_\ell \rangle \cdot O((\frac{\delta}{\ell})^{4/3})$$

which is much smaller.

On the other hand, even in this strengthened form, the nonlocal contributions decay only as a small power-law of the scale-ratios δ/ℓ and ℓ/Δ. This fact led Kraichnan to remark, for the similar wave-number ratio, that

> "However, the dependence on k/k' is not particularly strong, and thus the cascade is rather diffuse." –R. H. Kraichnan, J. Fluid Mech. **5** 497 (1959)

Similarly, T&L remarked

> "..we should not expect too much from the cascade model. After all, it is a very leaky cascade if half the water crossing a given level comes directly from all other pools uphill." – Tennekes & Lumley (1972), p. 261.

As these quotations reflect, a very large number of cascade steps is required before local interactions really dominate. In practice, a substantial fraction of the the energy transfer often comes from very non-local interactions.

<u>Local Time-Scales</u>

Now is a good time to discuss in more detail the time-scales t_ℓ at length-scale ℓ, or the <u>local eddy-turnover time</u>

$$t_\ell = \frac{\ell}{\delta v(\ell)}.$$

We have already argued in several places earlier that this is the time-scale for <u>Lagrangian evolution</u> at length-scale ℓ, i.e. that $\bar{D}_{\ell,t} \sim 1/t_\ell$. It is usually regarded as the time-scale for any "eddy" of size ℓ to change by $O(1)$ or to "turnover". It is also the time-scale set by the strain-rate at length-scale ℓ

$$\bar{\mathbf{S}}_\ell = O(\frac{\delta v(\ell)}{\ell}) = O(\frac{1}{t_\ell})$$

Thus, it is the time-scale in which structures of size ℓ are deformed by the fluid shears.

If the velocity field is Hölder continuous with exponent h, $0 < h < 1$, then

$$\delta v(\ell) \sim u_{rms}(\frac{\ell}{L})^h$$

and

$$t_\ell \sim t_L(\frac{\ell}{L})^{1-h}$$

with $t_L = L/u_{rms}$. We see that, as long as $h < 1$, then

$$t_\ell \to 0 \quad \text{for} \quad \ell \ll L.$$

Thus, the small-scale "eddies" evolve faster as ℓ decreases! For example, in K41 theory,

$$t_\ell \sim t_L(\frac{\ell}{L})^{2/3} \sim \langle\varepsilon\rangle^{-1/3}\ell^{2/3}$$

The energy cascade is thus an accelerated cascade, with each successive step sped up. Taking $\ell_n = 2^{-n}L$ for the length-scale of the nth step, then

$$t_n = t_L(\frac{\ell_n}{L})^{1-h} = t_L 2^{-(1-h)n}.$$

This has the remarkable property that

$$T = \sum_{n=0}^{\infty} t_n = t_L \sum_{n=0}^{\infty} 2^{-(1-h)n} < +\infty$$

for any $0 < h < 1$. Thus, the time T that it takes to make an infinite number of cascade steps is finite! This observation was first made by L.Onsager (1945,1949). It has several important implications.

First, t_ℓ is the time that it takes to transfer an $O(1)$ amount of energy across the length-scale ℓ. Indeed,

$$\Pi_\ell = O(\frac{1}{t_\ell} \cdot \delta v^2(\ell)).$$

Thus, the acceleration of the cascade is important to explain the observed dissipation of energy. In reality, there are only a finite number N of cascade steps to reach the dissipation range. For example, in K41 theory

$$N = \log_2(L/\eta) = \frac{3}{4}\log_2(Re).$$

The time for an $O(1)$ amount of energy to reach the dissipation range is then

$$T_N = \sum_{n=0}^{N} t_n$$

and $T_n \to T < +\infty$ in the limit $Re \to \infty$. Thus, turbulence can dissipate an $O(1)$ amount of energy in a time which is independent of Re, for $Re \gg 1$!

Another important consequence of "acceleration" of the cascade, is that it makes more plausible the hypothesis of universality. As the small-scales are evolving so quickly, the very large scales — which are non-universal — appear essentially "frozen". Furthermore, because of scale locality, there is no direct contact or communication between the largest scales and the small scales. Excitation is transferred only by a chain of chaotic intermediate scales. Thus, the small scales have plenty of time to approach an invariant distribution for fixed input (energy flux) from large-scales. Except for the conserved fluxes, which may vary slowly in time, all other information of the large scales is lost in the course of the cascade.

The chaotic dynamics of the turbulent flow at length-scale ℓ can be measured by the Lyapunov exponent λ_ℓ, which has units of inverse time and which gives the exponential rate of perturbation growth at that scale. Put another way, $1/\lambda_\ell$ is the "e-folding time" of perturbations at length-scale ℓ and may be expected to be approximately the same as the eddy-turnover time t_ℓ inside the inertial range $L \gg \ell \gg \eta$. At length-scales $\ell \ll \eta$, the Kolmogorov length-scale, viscosity damps out perturbations so that one expects a negative exponent $\lambda_\ell \approx -\nu/\ell^2$ and non-chaotic dynamics. These expectations are hard to prove mathematically for Navier-Stokes, although there has been some important progress for stochastically forced models:

> J. Bedrossian and S. Punshon-Smith. "Chaos in stochastic 2D Galerkin-Navier-Stokes," arXiv preprint arXiv:2106.13748 (2021).

For some toy "shell models" of the Navier-Stokes equation, these expected behaviors of Lyapunov exponents have been verified by numerical simulations:

> M. Yamada and Y. Saiki, "Chaotic properties of a fully developed model turbulence," Nonlin. Proc. Geophys. **14** 631-640 (2007).

Motivated by such considerations, the mathematical physicist David Ruelle in the following paper,

D. Ruelle, "Microscopic fluctuations and turbulence," Phys. Lett. **72A** 81–82 (1979),

asked the interesting question: how long will it take for *thermal fluctuations* at length-scale ℓ to be amplified to a macroscopic size? The largest positive Lyapunov should occur for $\ell \sim \eta$, the Kolmogorov length, with magnitude $1/t_\eta$ for $t_\eta = \eta^{2/3}/\varepsilon^{1/3} = (\nu/\varepsilon)^{1/2}$ the so-called Kolmogorov time. Thus, Ruelle asked what time t will it take for an initial thermal velocity fluctuation v'_η at scale η to grow exponentially according to the equation

$$e^{t/t_\eta} v'_\eta \simeq v_\eta$$

to a magnitude of order the Kolmogorov velocity $v_\eta = (\varepsilon\eta)^{1/3} = (\varepsilon\nu)^{1/4}$, which characterizes the magnitude of the turbulent velocity fluctuations at length-scale $\ell \simeq \eta$. To estimate the size of the initial thermal velocity fluctuation v'_ℓ at length-scale ℓ, Ruelle appealed to the Central Limit Theorem, which gives

$$v'_\ell \simeq \frac{v_{th}}{\sqrt{n\ell^d}},$$

with $v_{th} = (k_B T/m)^{1/2}$ the thermal velocity (which is of order the speed of sound c_s) and n the particle density, so that $N = n\ell^d$ represents the total number of particles in the region of radius ℓ. Here we recall that the N molecules of the fluid in a small spatial region with temperature T have their velocities distributed according to independent Maxwellian's:

$$P(\mathbf{v}_1, ..., \mathbf{v}_N) = \prod_{n=1}^{N} \left(\frac{m}{2\pi k_B T}\right)^{d/2} \exp\left(-\frac{m|\mathbf{v}_n - \mathbf{u}|^2}{2k_B T}\right)$$

If $\delta v(\ell)$ is the characteristic turbulent velocity at length-scale ℓ, then notice that

$$\frac{v'_\ell}{\delta v(\ell)} \simeq \left(\frac{k_B T}{\rho(\delta v(\ell))^2 \ell^d}\right)^{1/2} := \theta_\ell^{1/2}$$

and the term on the right is the small parameter which appears as the amplitude of the thermal noise term in the fluctuating Navier-Stokes equation at length-scale ℓ. As expected, the thermal fluctuations are negligible for large ℓ but grow as ℓ decreases.

Putting together these various estimates, Ruelle found that for $\ell \simeq \eta$, the time t for the thermal perturbation v'_η to grow to order the Kolmogorov velocity v_η is, for $d = 3$,

$$t \simeq t_\ell \ln(1/\theta_\eta), \quad \theta_\eta = \frac{k_B T}{\rho v_\eta^2 \eta^3}.$$

Although one expects a large value of $1/\theta_\eta \gg 1$, this quantity appears only inside a logarithm so that the growth time t is only a modest multiple of t_η! For example, consider some values of physical constants typical of the turbulent atmospheric boundary layer

$$T = 300^\circ \text{ K}, \quad \rho = 1.2 \text{ gm/cm}^3, \quad \nu = 0.15 \text{ cm}^2/\text{sec}, \quad \varepsilon = 400 \text{ cm}^3/\text{sec}^3.$$

Then

$$\eta = 0.54 \text{ mm}, \quad t_\eta = 19.4 \text{ msec}, \quad v_\eta = 2.78 \text{ cm/sec}$$

and with $k_B \doteq 1.38 \times^{-16}$ erg/K,

$$\theta_\eta = 2.83 \times 10^{-8}, \quad t \doteq (14.4) t_\eta.$$

More generally, one finds for typical terrestrial turbulent flows that the growth time for thermal perturbations at the Kolmogorov scale η to reach macroscopic magnitude is only $t \simeq 5 - 15 t_\eta$, i.e. just a few Kolmogorov times.

Once the perturbations have grown to macroscopic size at scale $\ell \simeq \eta$, they will infect the dynamics at the next larger scale $\ell \simeq 2\eta$ and produce errors of the size of the turbulent fluctuations at that twice larger scale, then $\ell \simeq 4\eta$, and so forth. Thus, one can expect an <u>inverse cascade of errors</u>. The expected time to double the length-scale of the error from ℓ_{n-1} to ℓ_n is just the turnover-time at that scale, or $t_n = t_{\ell_n}$. The total time T_n that it takes for a thermal perturbation to grow from the Kolmogorov scale η to length-scale ℓ_n is just

$$T_n = \sum_{m=n}^{N} t_m \simeq A t_n,$$

with A a constant of order unity at high Re with $N = \frac{3}{4} \log_2(Re) \gg 1$. This basic picture and the above formula for the growth time T_n were obtained in a fundamental paper:

E. N. Lorenz, "The predictability of a flow which possesses many scales of motion,"' Tellus 21, 289-307 (1969)

which we shall discuss in more depth later. These predictions were verified using a spectral closure (the "test-field model") in the following paper:

C. Leith and R. H. Kraichnan, "Predictability of turbulent flows," J. Atmos. Sci. **29** 1041–1058 (1972)

for both 2D and 3D space. Note that there is considerable difference in the physics of incompressible fluid turbulence in 2D and 3D, with energy cascading to large scales in 2D! However, the above paper found the above formula for growth time of errors to hold in the energy cascade range in both cases, with $A \doteq 10$ for 3D and $A \doteq 2.5$ for 2D. The papers

G. Boffetta and S. Musacchio, "Predictability of the inverse energy cascade in 2D turbulence," Phys. Fluids **13** 1060–1062 (2001)

G. Boffetta and S. Musacchio, "Chaos and predictability of homogeneous-isotropic turbulence,"' PRL 119, 054102 (2017)

verified the closure predictions for 2D and 3D turbulence via numerical simulation.

All of the above considerations imply, remarkably, that two different flows with *precisely* the same macroscopic initial velocity but with different realizations of thermal noise will lead to completely different velocity fields at all length-scales in the inertial-range within about one large-eddy turnover time! This suggests an intrinsic unpredictability of turbulent flows, with radically different solutions for the same macroscopic initial data.

<u>Thermal Energy Spectrum</u>

This is a suitable point to remark on the spectral interpretation of the result $v'_\ell = v_{th}/\sqrt{n\ell^3}$ for the thermal velocity fluctuation in space dimension $d = 3$. This formula corresponds to an energy spectrum of velocity fluctuations

$$E(k) = \frac{3}{2}\frac{k_B T}{\rho}\frac{4\pi k^2}{(2\pi)^3}.$$

Apart from the specific numerical prefactor, the result is obviously the correct one to yield

$$\langle |\overline{\mathbf{v}}'_\ell|^2\rangle = \int_0^\infty |\hat{G}_\ell(k)|^2 E(k)\,dk \simeq (\text{const.})\int_0^{1/\ell} E(k)\,dk \simeq (\text{const.})\frac{k_B T}{\rho \ell^3}$$

with $\overline{\mathbf{v}}'_\ell := \overline{\mathbf{v}}_\ell - \langle \overline{\mathbf{v}}_\ell\rangle$. The specific prefactor is found by using the equipartition of modal energies $\frac{1}{2}\rho|\widehat{\mathbf{v}}(\mathbf{k})|^2$ with thermal equilibrium value $\frac{1}{2}k_B T$ for each Fourier mode $\mathbf{k}$ and velocity component $i = 1, 2, 3$ and then counting the number of modes. This spectrum occurs for *any* fluid in local thermodynamic equilibrium and it has been observed in fluids at rest by light-scattering experiments. For a good review, see:

> José M. Ortiz de Zárate & Jan V. Sengers, *Hydrodynamic Fluctuations in Fluids and Fluid Mixtures* (Elsevier, 2006)

The fluctuating hydrodynamics of Landau & Lifschitz (1959), discussed in Chapter I, was proposed precisely to account for such thermal fluctuations. As we saw, this equation for an incompressible fluid takes the form

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p + \nu_\Lambda \triangle \mathbf{u} + \mathbf{f}$$

where p is chosen to enforce $\boldsymbol{\nabla}\cdot\mathbf{u} = 0$ and $f_i(\mathbf{x}, t)$ is a Gaussian space-time white-noise random process with mean zero. The covariance prescribed by the fluctuation-dissipation relation

$$\langle f_i(\mathbf{x}, t) f_j(\mathbf{x}', t')\rangle = \frac{2\nu_\Lambda k_B T}{\rho}\delta_{ij}\triangle_x \delta^d_\Lambda(\mathbf{x} - \mathbf{x}')\delta(t - t')$$

was chosen precisely to recover the equipartition energy spectrum

$$E(k) = \frac{k_B T}{\rho}\frac{4\pi k^2}{(2\pi)^3} \tag{$*$}$$

of the incompressible fluid modes, satisfying $\mathbf{k}\cdot\widehat{\mathbf{v}}(\mathbf{k}) = 0$. For a careful derivation, see:

G. Eyink et al., "Dissipation-range fluid turbulence and thermal noise," PRE (to appear, 2022), arXiv:2107.13954: `https://arxiv.org/abs/2107.13954`

The derived energy spectrum (*) differs from the previous result by the prefactor $\frac{3}{2} \to 1$, because the incompressibility constraint eliminates one degree of freedom per wavenumber. The equipartition spectrum of the longitudinal modes (sound waves) must be physically present, but these modes are expected to be dynamically decoupled from the transverse/solenoidal modes:

A. Donev et al., "Low Mach number fluctuating hydrodynamics of diffusively mixing fluids," Comm. App. Math. Comp. Sci. 9, 47–105 (2014)

Nothing that is stated above is special to turbulent flows and holds equally for laminar flows. There are, however, important implications for turbulent flows, where thermal and turbulent velocity fluctuations can compete. It has generally been expected that the turbulent spectrum in the dissipation range at scales (sufficiently) below the Kolmogorov length η should exhibit an exponential decay:

$$E(k) \sim v_\eta^2 \eta \exp(-\beta k\eta), \quad k\eta \gtrsim 1,$$

valid down to nearly $k\lambda_{mfp} \simeq 1$. For example, see

S. Khurshid, D. A. Donzis, and K. R. Sreenivasan, "Energy spectrum in the dissipation range," Phys. Rev. Fluids 3, 082601R (2018)

and many references therein. However, this conventional expectation is inconsistent with the thermal spectrum (*)! In fact, equating the two spectra allows us to identify the crossover wavenumber k_c as

$$k_c\eta = (2/\beta)W(\beta/2\theta_\eta^{1/2}) \simeq (2/\beta)\ln(\beta/2\theta_\eta^{1/2})$$

in terms of the Lambert W-function. Because of the slow logarithmic dependence, $k_c\eta \simeq 1$ even when $\theta_\eta \ll 1$. For example, in the ABL with $\theta_\eta = 2.83 \times 10^{-8}$ and taking $\beta \simeq 7$ (in agreement with Khurshid et al., 2018), one finds that $k_c\eta \doteq 2.25$. The first person, apparently, to point out the relevance of thermal fluctuations to the turbulent dissipation range was Robert Betchov:

R. Betchov, " On the fine structure of turbulent flows," J. Fluid Mech. 3, 205216 (1957); R. Betchov, "Thermal agitation and turbulence," in: *Rarefied Gas Dynamics*, Proceedings of the Second International Symposium, University of California, Berkeley, CA, 1960. L. Talbot, Ed. (New York, Academic Press, 1961), pp. 307-321.

However, only recently were Betchov's ideas given empirical support, first by Eyink et al. (2021) using a stochastic shell-model simulation, and subsequently by

J. B. Bell et al. "Thermal fluctuations in the dissipation range of homogeneous isotropic turbulence," J. Fluid Mech. 939: A12 (2022), `https://doi.org/10.1017/jfm.2022.188`

who numerically simulated homogeneous forced turbulence in a liquid water-glycerol mixture using incompressible fluctuating hydrodynamics. Their simulation was in periodic domain of size $5\,\mathrm{cm}^3$ at room temperature ($T = 300°$ K), with a mean dissipation per mass $\varepsilon = 17.7\ \mathrm{cm}^2/\mathrm{s}^3$, Kolmogorov length $\eta = 0.025$ cm, and Reynolds number Re=554 ($\mathrm{Re}_\lambda = 61$). See below:

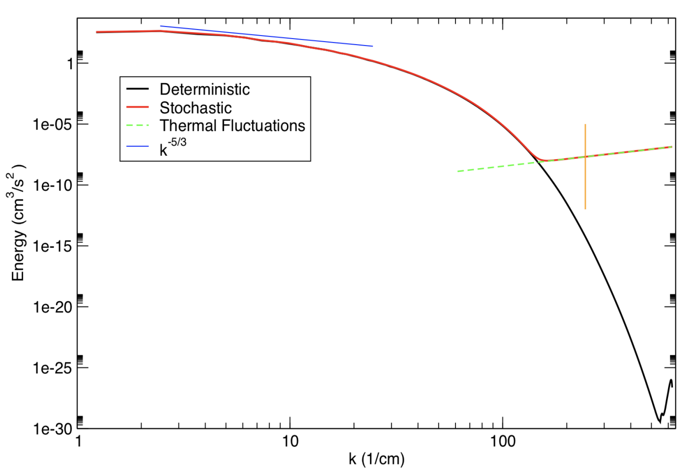


Figure 1: Specific energy density spectrum for Case 1 with thermal noise (red) and without (black). Blue line is $-5/3$ slope; green line represents the spectrum of thermal fluctuations given by Eq. (2.10). The Kolmogorov wavenumber $k_\eta = 2\pi/\eta \approx 251\ \mathrm{cm}^{-1}$ is indicated by the vertical orange line.

Turbulent Energy Spectrum, from Bell et al. (2022).

Thermal fluctuations thus become relevant near η, generally of order millimeters.

A well-known effect of such thermal fluctuations is renormalization of transport coefficients, such as viscosity, so that they become dependent on the wavenumber cut-off Λ (and also the frequency cutoff Ω). For example, see

> D. Forster, D. R. Nelson, and M. J. Stephen, "Large-distance and long-time properties of a randomly stirred fluid," Phys. Rev. A **16** 732–749 (1977)

who discussed this phenomenon by means of renormalization group methods. Note that we already indicated above in writing out the fluctuating Navier-Stokes equations that the viscosity ν_Λ is, in fact, Λ-dependent, just as the turbulent stress tensor $\boldsymbol{\tau}_\ell$ is ℓ-dependent! Thus, the usual picture that there is "scale-separation" in molecular dynamics is not true, because of the effects of thermal fluctuations, and there is more similarity between "molecular viscosity" and "turbulent viscosity" than is generally understood.

Nevertheless, essential differences remain. Indeed, the description of the turbulent stress $\boldsymbol{\tau}_\ell$ by means of an "eddy viscosity" is not accurate (as we shall discuss in more detail later). Furthermore, the thermal fluctuations are always weakly nonlinearly coupled, or "low Reynolds number". This may be seen by forming the "thermal Reynolds number" at every scale ℓ by

$$Re_\ell^\theta := \frac{\ell v'_{\theta,\ell}}{\nu_\ell} \sim \frac{\ell}{\nu_\ell}\frac{v_{th}}{\sqrt{n\ell^d}}$$

If the "bare viscosity" ν_0 at scales near $\ell \sim \lambda_{mfp}$, before renormalization by thermal fluctuations, can be estimated by $\nu_0 \sim v_{th}\lambda_{mfp}$, then we find using $\lambda_{intp} := n^{-1/d}$ that

$$Re_\ell^\theta \sim \left(\frac{\lambda_{intp}^d}{\lambda_{mfp}^2 \ell^{d-2}}\right)^{1/2} \lesssim 1 \qquad (**)$$

for $\ell \gtrsim (\lambda_{intp}^d/\lambda_{mfp}^2)^{1/(d-2)}$. Recall $\lambda_{mfp} \gg \lambda_{intp}$ for a low-density gas, whereas $\lambda_{mfp} \simeq \lambda_{intp}$ for a liquid. Thus, (**) holds down to $\ell \simeq \lambda_{mfp}$, where the hydrodynamic approximation breaks down. In particular, the "UV strong coupling regime" for $d > 2$ discussed by Forster et al. (1977) is never achieved in molecular fluids.

These considerations do, however, raise an important issue about our earlier discussion of kinetic energy balance for turbulent fluid flows, which was based entirely on the deterministic Navier-Stokes equations. How does the existence of thermal noise alter that calculation? It is straightforward to derive the kinetic energy balance for 3D fluctuating Navier-Stokes equation:

$$\partial_t\left(\frac{1}{2}|\mathbf{v}|^2\right)+\boldsymbol{\nabla}\cdot\left[\left(\frac{1}{2}|\mathbf{v}|^2+p\right)\mathbf{v}-\nu_\Lambda\boldsymbol{\nabla}\left(\frac{1}{2}|\mathbf{v}|^2\right)\right]=-\nu_\Lambda|\boldsymbol{\nabla}\mathbf{v}|^2+\mathbf{v}\cdot\mathbf{f}+\frac{2\nu_\Lambda k_BT}{\rho}\cdot\frac{1}{L^3}\sum_{|\mathbf{k}|<\Lambda}|\mathbf{k}|^2,$$

where L is the side-length of the periodic flow domain. Two new terms arise from the stochastic forcing, the obvious term $\mathbf{v}\cdot\mathbf{f}$ which has vanishing mean and the final constant term which arises from the Itō chain rule. Note that in the statistical steady-state (fluid entirely at rest), one obtains from this equation the mean energy balance

$$\nu_\Lambda\langle|\boldsymbol{\nabla}\mathbf{v}_\Lambda|^2\rangle=2\nu_\Lambda\int_0^\Lambda dk\,k^2E(k), \qquad (\star)$$

where $E(k)$ is the thermal equipartition spectrum (*) and where now $\mathbf{v}_\Lambda$ denotes the velocity in the model with UV cutoff Λ. It follows that in thermal equilibrium, the gradients become strongly Λ-dependent with $\langle|\boldsymbol{\nabla}\mathbf{v}_\Lambda|^2\rangle\propto\Lambda^5$. As we can see from the simulation of Bell et al. (2021), however, the same equipartition spectrum should occur in a turbulent flow, e.g. decaying turbulence behind a grid. Thus the same divergence with Λ must occur in turbulent flows! How is it possible then to use deterministic Navier-Stokes to analyze the kinetic energy balance?

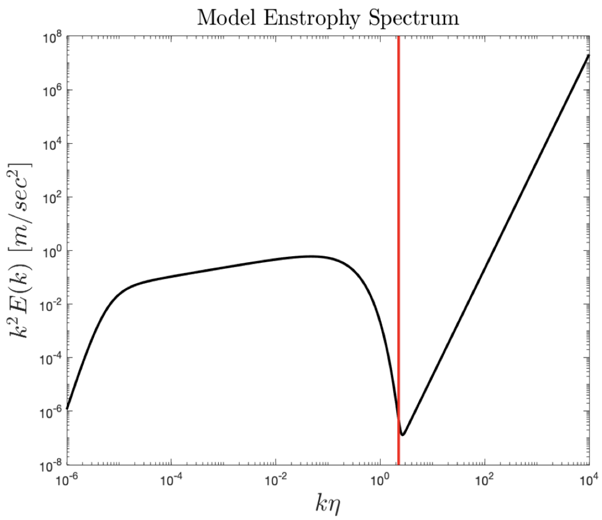
To answer this question, we will study the magnitude of the gradients $\boldsymbol{\nabla}\mathbf{v}_\Lambda$ by means of a model turbulent energy spectrum

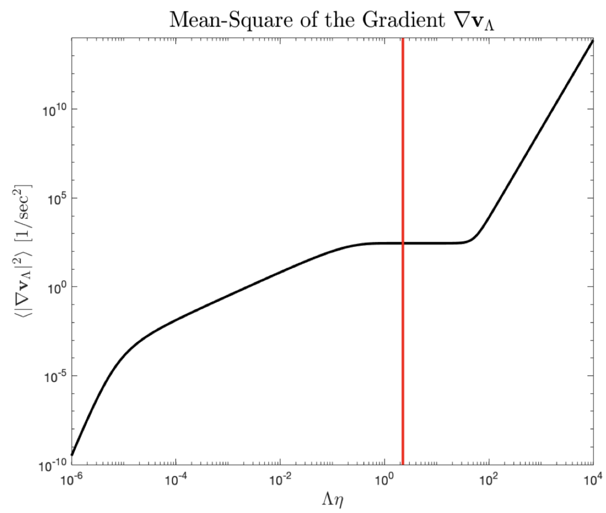
$$E(k)=C_K(\varepsilon L)^{2/3}\frac{L^5k^4}{(1+(kL)^2)^{17/6}}\exp(-bk\eta)+\frac{k_BT}{\rho}\frac{4\pi}{(2\pi)^3}k^2$$

that was proposed by

T. von Kármán, "Progress in the statistical theory of turbulence," Proc. Nat. Acad. Sci. 34, 530 (1948)

to which we have added the thermal equipartition spectrum. In the following figure we plot the enstrophy spectrum $k^2E(k)$ obtained from this model and also the mean-square gradients $\langle|\boldsymbol{\nabla}\mathbf{v}_\Lambda|^2\rangle$ obtained from $(\star)$ by integration over k:





The parameters here were chosen to correspond to the atmospheric boundary layer, as discussed earlier. The vertical red line marks the position of the crossover wavenumber $k_c\eta$, where it can be seen that $k^2E(k)$ has a deep local minimum, after the exponential decay around the Kolmogorov scale but before the k^2 growth of the equipartition spectrum. Because of this deep minimum, the mean-square gradient $\langle|\boldsymbol{\nabla}\mathbf{v}_\Lambda|^2\rangle$ obtained by the integration $(\star)$ is almost independent of Λ within a 3-decade interval around k_c. For any Λ chosen in this range it follows that

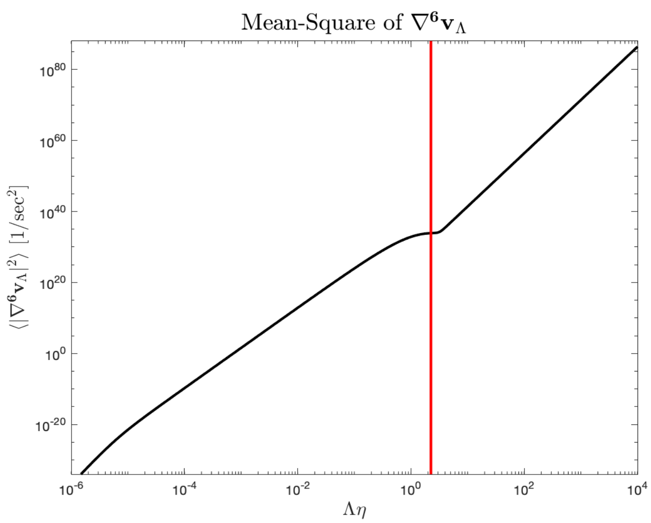
$$|\boldsymbol{\nabla}\mathbf{v}_\Lambda|^2 \gg \frac{2k_BT}{\rho}\cdot\frac{1}{L^3}\sum_{|\mathbf{k}|<\Lambda}|\mathbf{k}|^2,$$

and thus the kinetic energy balance for fluctuating Navier-Stokes becomes

$$\partial_t\left(\frac{1}{2}|\mathbf{v}_\Lambda|^2\right)+\boldsymbol{\nabla}\cdot\left[\left(\frac{1}{2}|\mathbf{v}_\Lambda|^2+p_\Lambda\right)\mathbf{v}_\Lambda-\nu_\Lambda\boldsymbol{\nabla}\left(\frac{1}{2}|\mathbf{v}_\Lambda|^2\right)\right]\doteq-\nu_\Lambda|\boldsymbol{\nabla}\mathbf{v}_\Lambda|^2,$$

where we have neglected also the term $\mathbf{v}\cdot\mathbf{f}$, which is small and has mean zero. It follows that the energy balance reduces approximately to that for deterministic Navier-Stokes in this range of Λ. We may refer to the velocity-gradients $\boldsymbol{\nabla}\mathbf{v}_\Lambda$ in this interval around $\Lambda\simeq k_c$ as <u>effective gradients</u>, because these are the quantities which would be measured with any space-resolution $1/\Lambda$ selected in this range. Likewise, ν_Λ is the viscosity which would be measured experimentally in a laminar flow, slightly renormalized from its "bare value" ν_0 by thermal fluctuations. Thus, the validity of the energy balance for deterministic Navier-Stokes is justified to a good approximation.

Note, however, that not all of the predictions of deterministic Navier-Stokes are correct! For example, the following plot shows the mean-square of the 6th-order space-derivative $\langle|\nabla^6\mathbf{v}_\Lambda|^2\rangle = 2\int_0^\Lambda dk\, k^{12}E(k)$ calculated from the von Kármán model spectrum:



As can be seen, there is no range of Λ for which $\langle|\nabla^6\mathbf{v}_\Lambda|^2\rangle$ is Λ-independent and the predictions of deterministic Navier-Stokes are not accurate for any choice of Λ.

Helicity Cascade

However, kinetic energy is not the only ideal invariant of 3D Euler! There is also the helicity

$$H = \int d^3\mathbf{x}\ \mathbf{v}(\mathbf{x})\cdot\boldsymbol{\omega}(\mathbf{x}).$$

It was conjectured by

A. Brissand, U. Frisch, J. Leorat, M. Lesieur, and A. Mazure, "Helicity cascades in fully developed isotropic turbulence," Phys. Fluids **16**, 1366-1368 (1973) that flows with large-scale helicity (either by forcing or initial) shall have a *joint cascade of energy & helicity* to small-scales, i.e. a helicity cascade coexisting with the energy cascade.

The large-scale helicity balance can be derived from the coarse-grained Navier-Stokes equation in the form

$$\partial_t \bar{\mathbf{v}}_\ell = \bar{\mathbf{v}}_\ell \times \bar{\boldsymbol{\omega}}_\ell - \boldsymbol{\nabla}(\bar{p}_\ell + \tfrac{1}{2}|\bar{\mathbf{v}}_\ell|^2) + \mathbf{f}_\ell^s + \nu \triangle \bar{\mathbf{v}}_\ell$$

with $\mathbf{f}_\ell^s = -\boldsymbol{\nabla} \cdot \boldsymbol{\tau}_\ell$ the subscale force; $\bar{e}_\ell = \frac{1}{2}|\bar{\mathbf{v}}_\ell|^2$. Taking the curl of both sides gives the coarse-grained vorticity equation

$$\partial_t \bar{\boldsymbol{\omega}}_\ell = \boldsymbol{\nabla} \times (\bar{\mathbf{v}}_\ell \times \bar{\boldsymbol{\omega}}_\ell + \mathbf{f}_\ell^s) + \nu \triangle \bar{\boldsymbol{\omega}}_\ell$$

From this it is easy to derive that the large-scale helicity density $\bar{h}_\ell = \bar{\mathbf{v}}_\ell \cdot \bar{\boldsymbol{\omega}}_\ell$ satisfies

$$\partial_t \bar{h}_\ell + \boldsymbol{\nabla} \cdot \bar{\mathbf{J}}_\ell^H = -\Lambda_\ell - 2\nu \boldsymbol{\nabla} \bar{\mathbf{v}}_\ell : \boldsymbol{\nabla} \bar{\boldsymbol{\omega}}_\ell$$

where

$$\begin{aligned}
\bar{\mathbf{J}}_\ell^H &= \bar{h}_\ell \bar{\mathbf{v}}_\ell + (\bar{p}_\ell - \bar{e}_\ell)\bar{\boldsymbol{\omega}}_\ell + \bar{\mathbf{v}}_\ell \times \mathbf{f}_\ell^s - \nu \boldsymbol{\nabla} \bar{h}_\ell \\
&= \text{ space transport of large-scale helicity} \\
2\nu \boldsymbol{\nabla} \bar{\mathbf{v}}_\ell : \boldsymbol{\nabla} \bar{\boldsymbol{\omega}}_\ell &= \text{ viscous dissipation of helicity} \\
\Lambda_\ell &= -2\bar{\boldsymbol{\omega}}_\ell \cdot \mathbf{f}_\ell^s = \text{ helicity flux}
\end{aligned}$$

The latter quantity transfers helicity between scales. It is easy to see how this term does so if one recalls the topological meaning of large-scale helicity:

$$H_\ell = \textstyle\int d^3\mathbf{x}\ \bar{\mathbf{v}}_\ell(\mathbf{x}) \cdot \bar{\boldsymbol{\omega}}_\ell(\mathbf{x}),$$

which gives the asymptotic linking number of the lines of $\bar{\boldsymbol{\omega}}_\ell$, i.e. the flux of large-scale vorticity through the closed lines of the large-scale vorticity itself. This was proved by V.I.Arnold, Sel. Math.Sov. **5**, 327 (1986); see also Arnold & Khesin (1998). Thus, we can understand that it is the parallel component of $\mathbf{f}_\ell^s$, along the lines of $\bar{\boldsymbol{\omega}}_\ell$, which modifies the large-scale helicity. The component of the turbulent force $\mathbf{f}_\ell^s$ parallel to $\bar{\boldsymbol{\omega}}_\ell$ accelerated fluid about closed loops of $\bar{\boldsymbol{\omega}}_\ell$-lines, generating circulation around them. Vorticity flux is this created/destroyed though the vortex-loop:

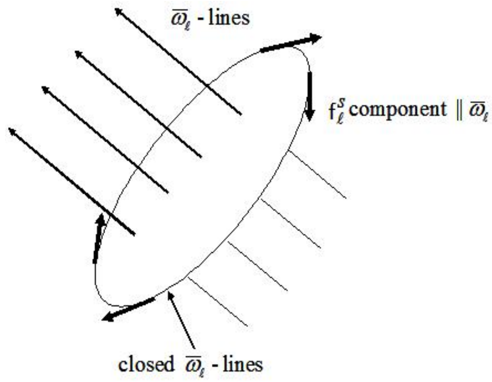


FIGURE: A VORTEX LOOP.

Note that

$$2\nu \boldsymbol{\nabla}\bar{\mathbf{v}}_\ell : \boldsymbol{\nabla}\bar{\boldsymbol{\omega}}_\ell = O(\nu \tfrac{\delta v^2(\ell)}{\ell^3})$$

whereas

$$\Lambda_\ell = O(\tfrac{\delta v^3(\ell)}{\ell^2}).$$

Thus the viscous destruction of helicity is certainly negligible in the limit as $\nu \to 0$ with ℓ fixed.

If $\mathbf{v}$ is Hölder continuous with exponent $0 < h < 1$, then

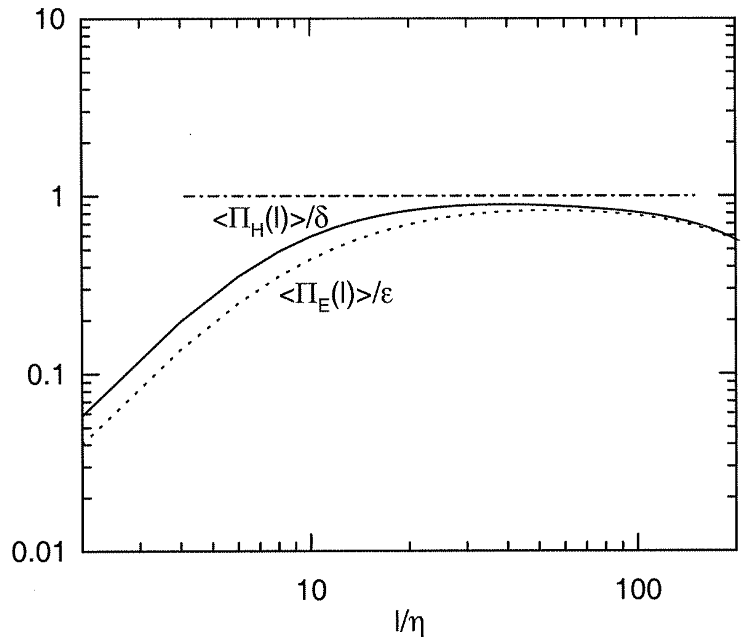
$$\Lambda_\ell = O(\ell^{3h-2})$$

as an upper bound. Thus, a non-vanishing Λ_ℓ for $\ell \to 0$ is possible with any $h \leq \frac{2}{3}$ and, in particular, for the K41 value $h = \frac{1}{3}$. It is quite possible to have co-existing cascade of energy & helicity! Note that, *a priori*

$$\Lambda_\ell = \Pi_\ell \cdot O(1/\ell)$$

However, these are only upper bounds and both Λ_ℓ (and Π_ℓ) take on both positive and negative values and can have significant cancellations in averages over space or time. These cancellations are disccussed in more detail by

Q. Chen, S.Chen & G. L. Eyink, "The joint cascade of energy and helicity in three-dimensional turbulence,", Phys. Fluids **15**, 361-374(2003)



Space-time averages of energy flux and helicity flux versus filter length. δ = helicity input, ε = energy input. Data from the 512^3 DNS of Chen et al. (2003) of forced homogeneous, isotropic turbulence.

The role of helicity in turbulent flow is still rather mysterious. Note that H is a pseudoscalar (which changes sign under space-reflection) so that it can only be present for reflection-non-symmetric forcing and/or initial conditions, on average. Of course, there can still be local helicity — positive in some regions, negative in others — that vanishes on average. It has been suggested that high local helicity (of either sign) may be correlated with low local energy dissipation:

A. Tsinober & E. Levich, "On the helical nature of three-dimensional coherent structures in turbulent flows," Phys. Lett. A **99** 321-323 (1983)

but this does not seem borne out by simulations

M. M. Rogers & P. Moin, Phys. Fluids **30** 2622 (1987); R.M.Kerr, Phys. Rev. Lett. **59** 783 (1987)

and experiment

J. M. Wallace, J. -L. Balint and L. Ong, Phys. Fluids A **4** 2013 (1992)

For a general review, see

H. K. Moffat & A. Tsinober, "Helicity in laminar and turbulent flow," Annu. Rev. Fluid Mech. **24**, 281 (1992)

We shall return to helicity and related issues when we consider later in depth the vorticity dynamics in turbulent flows.

<u>Entropy Cascade</u>

If we consider the complete Navier-Stokes-Fourier system of equations governing the dynamics of an incompressible fluid

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = -\boldsymbol{\nabla}p + \nu\triangle\mathbf{v}, \quad \nabla\cdot\mathbf{v} = 0$$

$$\partial_t T + (\mathbf{v}\cdot\boldsymbol{\nabla})T = \lambda_T\triangle T + \varepsilon/c_P$$

with $\varepsilon = 2\nu|\mathbf{S}|^2$, then there is yet another ideal invariant, the total <u>thermodynamic entropy</u>: $S = \int d^3x\ s$ with

$$s = \rho c_P \ln T.$$

It is easy to check that, with thermal conductivity defined as $\kappa = \rho c_P \lambda_T$,

$$\partial_t s + \boldsymbol{\nabla}\cdot(s\mathbf{v} - \lambda_T\boldsymbol{\nabla}s) = \frac{\kappa|\boldsymbol{\nabla}T|^2}{T^2} + \rho\varepsilon \geq 0,$$

which is the local form of the second law of thermodynamics for an incompressible fluid. For smooth solutions of ideal Euler equations with $\nu = \kappa = 0$, entropy is thus conserved.

We have discussed extensive evidence that, in a turbulent flow, $\varepsilon \nrightarrow 0$ as $\nu \to 0$. This already suggests that entropy may, in fact, not be conserved for the ideal limit. In addition, it is possible that $\kappa|\boldsymbol{\nabla}T|^2/T^2 \nrightarrow 0$ as ν, $\kappa \to 0$ together, because of development of large temperature gradients. This possibility was suggested (in a special case of strong temperature forcing) by

> A. M. Obukhov, Temperature Field Structure in a Turbulent Flow, Izvestiia Akademii Nauk SSSR, Ser. Geogr. i Geofiz **13**, 58 (1949).

and more generally by

> G. L. Eyink and T. D. Drivas, Cascades and Dissipative Anomalies in Compressible Fluid Turbulence, Phys. Rev. X, **8**, 011022 (2018)

Both of these papers suggested that such anomalous entropy production may indeed occur, and be associated to a *cascade of entropy* from the small-scales where entropy is produced to the large-scales where entropy accumulates, in the form of a more spatially uniform and/or rising large-scale temperature. A statistical steady-state is possible as well, if excess entropy is removed by large-scale cooling or by a source of temperature inhomogeneities (e.g. differential heating/cooling) that sustains large-scale thermal structure.

A large-scale <u>entropy balance</u> can be introduced in an obvious way by considering

$$\underline{s}_\ell = \rho c_P \ln \overline{T}_\ell$$

as the measure of the <u>resolved/large-scale entropy</u>. In particular, note that $\underline{s}_\ell \geq \overline{s}_\ell$, because entropy is a concave function of temperature. Thus, the total entropy increases under coarse-graining, so that "myopic" observations at space resolution ℓ cannot miss anomalous entropy production. It is straightforward to show using the thermofluid equations that

$$\partial_t \underline{s}_\ell + \boldsymbol{\nabla} \cdot [\underline{s}_\ell \overline{\mathbf{v}}_\ell + \underline{\beta}_\ell \left(\tau_\ell(u, \mathbf{v}) - \kappa \boldsymbol{\nabla} \overline{T}_\ell\right)] = \boldsymbol{\nabla} \underline{\beta}_\ell \cdot \tau_\ell(u, \mathbf{v}) + \frac{\kappa |\boldsymbol{\nabla} \overline{T}_\ell|^2}{\overline{T}_\ell^2} + \rho \underline{\beta}_\ell \overline{\varepsilon}_\ell,$$

where $u = \rho c_P T$ is the internal energy per volume of the fluid and $\underline{\beta}_\ell = 1/\overline{T}_\ell$. The terms in this entropy balance proportional to κ vanish in the limit as κ, $\nu \to 0$ under reasonable assumptions (see Homework). However, the quantity $\rho\overline{\varepsilon_\ell}$ representing the viscous dissipation of kinetic energy is clearly not vanishing! This is the same term that appears in the balance equation of the unresolved/small-scale kinetic energy, which in the limit $\nu \to 0$ becomes

$$\partial_t \rho k_\ell + \boldsymbol{\nabla} \cdot \left[\rho k_\ell \overline{\mathbf{v}}_\ell + \tau_\ell(P,\mathbf{v}) + \frac{1}{2}\rho\tau_\ell(v_i,v_i,\mathbf{v})\right] = \rho\Pi_\ell - \rho\overline{\varepsilon}_\ell.$$

As discussed earlier, the coarse-grained viscous dissipation rate does not vanish as $\nu \to 0$. Thus, the balance equation for $\underline{s}$ as κ, $\nu \to 0$ does not involve only ideal dynamical terms, unlike the large-scale balances for kinetic energy and helicity.

There is, however, an alternative definition of "resolved/large-scale entropy" whose balance involves only ideal dynamics, which is given simply by

$$\underline{s}_\ell^* = \underline{s}_\ell + \underline{\beta}_\ell \rho k_\ell.$$

Because $k_\ell \geq 0$, it is again true that $\underline{s}_\ell^* \geq \overline{s}_\ell$ and furthermore $\lim_{\ell\to 0} \underline{s}_\ell^* = s$. Thus, the quantity $\underline{s}_\ell^*$ is a reasonable choice as a "resolved/large-scale entropy" (and it is furthermore shown in Eyink & Drivas (2016) that $\underline{s}_\ell^*$ is the entropy obtained from standard thermodynamic relations if the "resolved internal energy" is obtained from coarse-grained observations of the conserved energy and momentum). It is straightforward to show that the term $\rho\underline{\beta}_\ell\overline{\varepsilon}_\ell$ cancels in the balance equation for $\underline{s}_\ell^*$, which in the limit κ, $\nu \to 0$ becomes:

$$\partial_t \underline{s}_\ell^* + \boldsymbol{\nabla} \cdot [\underline{s}_\ell^* \overline{\mathbf{v}}_\ell + \underline{\beta}_\ell \mathbf{q}_\ell] = \rho\underline{\beta}_\ell\Pi_\ell + (\overline{D}_t\underline{\beta}_\ell)\rho k_\ell + \boldsymbol{\nabla}\underline{\beta}_\ell \cdot \mathbf{q}_\ell := \Sigma_\ell$$

where we have defined a "turbulent heat-transport vector"

$$\mathbf{q}_\ell := \tau_\ell(h,\mathbf{v}) + \frac{1}{2}\rho\tau_\ell(v_i,v_i,\mathbf{v})$$

with $h = u + P = \rho(c_p T + p)$ the thermodynamic enthalpy per volume. The expression Σ_ℓ on the righthand side of this resolved entropy balance does not depend upon κ or ν explicitly and represents an <u>entropy flux</u> from unresolved length scales $< \ell$ to resolved scales $> \ell$. Each of the three terms appearing in Σ_ℓ has a transparent physical interpretation:

$$\rho\underline{\beta}_\ell\Pi_\ell \quad = \quad \text{entropy production from turbulent energy cascade}$$

$$
\begin{aligned}
(\overline{D}_t\underline{\beta}_\ell)\rho k_\ell &= \text{entropy production from large-scale temperature change} \\
&\quad \text{anti-correlated with subscale kinetic energy density} \\
\nabla\underline{\beta}_\ell\cdot\mathbf{q}_\ell &= \text{entropy production from turbulent heat-transport} \\
&\quad \text{down the gradient of large-scale temperature}
\end{aligned}
$$

When the entropy production is non-vanishing in the limit $\kappa,\ \nu\to 0$ (anomalous), then now-standard arguments show that there must be a cascade of entropy. Increase of entropy at large-scales seen by a "myopic" observer cannot be accounted for by thermal-conductive or viscous entropy production, and require non-vanishing entropy flux. For example, if an initial large-scale temperature distribution is created in a turbulent flow, one expects the temperature field to become nearly homogenous at large-scales due to turbulent heat transport. This is the type of situation considered by Obukhov (1949), where decay of an initial temperature inhomogeneity is associated to cascade of entropy from small-scales up to the scale of the inhomogeneity. Another example is driven turbulence with a nearly homogeneous temperature, where the large-scale temperature must slowly increase due to viscous heating. Here also entropy must cascade from small-scales to account for the gradual increase in large-scale temperature.

Just as for cascades of kinetic energy and helicity, a non-vanishing flux of entropy requires "rough" or "non-smooth" fields of both velocity and temperature. By exploiting the above explicit expression Σ_ℓ for entropy flux, Eyink & Drivas (2018) derive constraints on the scaling exponents ζ_p^v of velocity and ζ_p^T of temperature for all $p\geq 3$ of the form

$$
2\zeta_p^T+\zeta_p^v\leq p,\quad \zeta_p^T+2\zeta_p^v\leq p,\quad 3\zeta_p^v\leq p,
$$

in order that a turbulent entropy cascade can be sustained.