(D) Estimating General Terms in the Effective Equations

We now return to the effective large-scale equations

$$\partial_t \bar{\mathbf{v}}_\ell + (\bar{\mathbf{v}}_\ell \cdot \boldsymbol{\nabla})\bar{\mathbf{v}}_\ell + \boldsymbol{\nabla} \cdot \boldsymbol{\tau}_\ell = -\boldsymbol{\nabla} \bar{p}_\ell + \nu \triangle \bar{\mathbf{v}}_\ell + \bar{\mathbf{f}}_\ell.$$

It is important to be able to estimate the magnitude of the various terms that appear here. We start with $\mathbf{f}_\ell$. Note that $\bar{\mathbf{f}}_\ell = \mathbf{f} - \mathbf{f}'_\ell$ and

$$\mathbf{f}'_\ell = -\int d^d r \; G_\ell(\mathbf{r})\delta\mathbf{f}(\mathbf{r})$$

$$\begin{aligned} \Longrightarrow |\mathbf{f}'_\ell| &\leq \int d^d r \; G_\ell(\mathbf{r})|\delta\mathbf{f}(\mathbf{r})| \\ &\leq (\sup_{\mathbf{x}} |\boldsymbol{\nabla}\mathbf{f}(\mathbf{x})|) \int d^d r \; G_\ell(\mathbf{r})|\mathbf{r}|, \text{ assuming } \mathbf{f} \text{ is smooth and Taylor expanding!} \\ &= C\ell \|\boldsymbol{\nabla}\mathbf{f}\|_\infty \text{ with } C = \int d^d\rho G(\rho)|\rho| \end{aligned}$$

Thus, with a smooth (large-scale) external force, we see that

$$\bar{\mathbf{f}}_\ell \cong \mathbf{f} \text{ when } \ell \ll L^f_\nabla = \frac{|\mathbf{f}|}{|\boldsymbol{\nabla}\mathbf{f}|} = \text{gradient-length of the force}$$

We have already considered $\nu \triangle \bar{\mathbf{v}}_\ell$, but we now obtain an improved estimate:

$$\nu \triangle \bar{\mathbf{v}}_\ell(\mathbf{x}) = -\frac{1}{\ell^2} \int d^d r \; (\triangle G)_\ell(\mathbf{r})\delta\mathbf{v}(\mathbf{r};\mathbf{x})$$

$$\Longrightarrow \text{(with compactly-supported filter)}$$

$$\nu \triangle \bar{\mathbf{v}}_\ell = O\left(\frac{\nu \delta v(\ell)}{\ell^2}\right)$$

Before we had $\nu \triangle \bar{\mathbf{v}}_\ell = O(\nu \|\mathbf{v}\|_2/\ell^2)$ but the above estimate is much smaller for $\eta \ll \ell \ll L$.

Now consider the subscale force $\mathbf{f}^s_\ell = -\boldsymbol{\nabla} \cdot \boldsymbol{\tau}_\ell$, using the identity

$$f^s_{\ell i} = \frac{1}{\ell}\left\{\int d^d r \; (\partial_j G)_\ell \delta v_i(\mathbf{r})\delta v_j(\mathbf{r}) - \int d^d r \; (\partial_j G)_\ell(\mathbf{r})\delta v_i(\mathbf{r}) \int d^d r' G_\ell(\mathbf{r}')\delta v_j(\mathbf{r}')\right\}$$

which is easily verified (using $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$). From this, it follows that

$$\mathbf{f}^s_\ell = O\left(\frac{\delta v^2(\ell)}{\ell}\right).$$

It is useful for later purposes to note a more general technique to derive such an identity for $\partial_k \tau_{ij}$, using the fact that

$$\begin{aligned} \tau_\ell(v_i,v_j)(\mathbf{x}+\mathbf{a},t) &= \int d^dr\ G_{\mathbf{a},\ell}(\mathbf{r})\delta v_i(\mathbf{r},t)\delta v_j(\mathbf{r},t) \\ &\quad -\int d^dr\ G_{\mathbf{a},\ell}(\mathbf{r})\delta v_i(\mathbf{r},t)\int d^dr' G_{\mathbf{a},\ell}(r')\delta v_j(\mathbf{r}',t) \qquad (*) \end{aligned}$$

with kernel $G_{\mathbf{a},\ell}$ centered at point $\mathbf{a}$:

$$G_{\mathbf{a},\ell}(\mathbf{r}) = \ell^{-d}G(\tfrac{\mathbf{r}-\mathbf{a}}{\ell}).$$

By expanding to 1st-order in $\mathbf{a}$, we derive

$$\begin{aligned} \partial_k\tau_\ell(v_i,v_j) &= -\frac{1}{\ell}\left\{\int d^dr(\partial_kG)_\ell(\mathbf{r})\delta v_i(\mathbf{r})\delta v_j(\mathbf{r}) \right. \\ &\quad -\int d^dr(\partial_kG)_\ell(\mathbf{r})\delta v_i(\mathbf{r})\int d^dr' G_\ell(r')\delta v_j(\mathbf{r}') \\ &\quad \left. -\int d^drG_\ell(\mathbf{r})\delta v_i(\mathbf{r})\int d^dr'(\partial_kG)_\ell(\mathbf{r}')\delta v_j(\mathbf{r}')\right\}. \end{aligned}$$

Summing over $k=j$ and using incompressibility $\boldsymbol{\nabla}\cdot\mathbf{v}=0$, then gives the identity on the previous page for $\mathbf{f}_\ell^s$.

We have only to justify the identity (*). We first note that by its definition

$$\begin{aligned} \tau_\ell(v_i,v_j)(\mathbf{x}+\mathbf{a}) &= \int d^dr\ G_\ell(\mathbf{r})[v_i(\mathbf{x}+\mathbf{a}+\mathbf{r})-v_i(\mathbf{x}+\mathbf{a})][v_j(\mathbf{x}+\mathbf{a}+\mathbf{r})-v_j(\mathbf{x}+\mathbf{a})] \\ &\quad -\int d^dr\ G_\ell(\mathbf{r})[v_i(\mathbf{x}+\mathbf{a}+\mathbf{r})-v_i(\mathbf{x}+\mathbf{a})]\int d^dr\ G_\ell(r)[v_j(\mathbf{x}+\mathbf{a}+\mathbf{r})-v_j(\mathbf{x}+\mathbf{a})] \end{aligned}$$

Making the change of variables $\mathbf{r}+\mathbf{a}\to\mathbf{r}$ gives

$$\begin{aligned} \tau_\ell(v_i,v_j)(\mathbf{x}+\mathbf{a}) &= \int d^dr\ G_{\mathbf{a},\ell}(\mathbf{r})[\delta v_i(\mathbf{r};\mathbf{x})-\delta v_i(\mathbf{a};\mathbf{x})][\delta v_j(\mathbf{r};\mathbf{x})-\delta v_j(\mathbf{a};\mathbf{x})] \\ &\quad -\int d^dr\ G_{\mathbf{a},\ell}(\mathbf{r})[\delta v_i(\mathbf{r};\mathbf{x})-\delta v_i(\mathbf{a};\mathbf{x})]\int d^dr\ G_{\mathbf{a},\ell}(r)[\delta v_j(\mathbf{r};\mathbf{x})-\delta v_j(\mathbf{a};\mathbf{x})], \end{aligned}$$

since

$$\begin{aligned} \mathbf{v}(\mathbf{x}+\mathbf{r})-\mathbf{v}(\mathbf{x}+\mathbf{a}) &= [\mathbf{v}(\mathbf{x}+\mathbf{r})-\mathbf{v}(\mathbf{x})]-[\mathbf{v}(\mathbf{x}+\mathbf{a})-\mathbf{v}(\mathbf{x})] \\ &= \delta\mathbf{v}(\mathbf{r};\mathbf{x})-\delta\mathbf{v}(\mathbf{a};\mathbf{x}). \end{aligned}$$

However, $\delta\mathbf{v}(\mathbf{a};\mathbf{x})$ does not depend upon $\mathbf{r}$ and is thus a "constant" with respect to the average over $\mathbf{r}$ with density $G_{\mathbf{a},\ell}(\mathbf{r})$. Since cumulants are invariant to shifts of the random variables by constants, this yields the formula (*). We shall use this "shift-trick" again later for higher-order terms.

We now see indeed that the viscous term is negligible with respect to the subscale force term, for ℓ such that

$$Re_\ell \equiv \frac{\delta v(\ell)\ell}{\nu} \gg 1.$$

Within K41 theory, this holds whenever $\ell \gg \eta$.

Now consider the large-scale pressure gradient $\boldsymbol{\nabla}\bar{p}_\ell$. It is very easy to derive

$$\boldsymbol{\nabla}\bar{p}_\ell = -\frac{1}{\ell}\int (\boldsymbol{\nabla}G)_\ell(\mathbf{r})\delta p(\mathbf{r})d^d r$$

from which

$$\boldsymbol{\nabla}\bar{p}_\ell = O(\frac{\delta p(\ell)}{\ell})$$

with $\delta p(\ell;\mathbf{x}) = \sup_{|\mathbf{r}|\leq\ell}|\delta p(\mathbf{r};\mathbf{x})|$. However, this is not so useful, since we do not know the magnitude of $\delta p(\ell)$! It is more instructive to use the relation

$$\triangle\bar{p}_\ell = -\frac{1}{\ell^2}\int d^d r\ (\partial_i\partial_j G)_\ell(\mathbf{r})\delta v_i(\mathbf{r})\delta v_j(\mathbf{r})$$

which follows by filtering the Poisson equation $-\triangle p = \partial_i\partial_j(v_i v_j)$. From this, we get

$$\triangle\bar{p}_\ell = O(\frac{\delta v^2(\ell)}{\ell^2}).$$

This suggests that

$$\delta p(\ell) = O(\delta v^2(\ell)) \qquad (\bigstar)$$

which would imply that

$$\boldsymbol{\nabla}\bar{p}_\ell = O(\frac{\delta v^2(\ell)}{\ell}),$$

and thus has the same order of magnitude as the subscale force term. In fact, it is possible to prove various rigorous forms of ($\bigstar$). This will involve a bit of mathematics more sophisticated than any that we have used up to now (or will later!) This is not surprising — the pressure is one of the most mysterious and challenging objects in the study of incompressible turbulence!

As an important example of a rigorous form of ($\bigstar$), we show that

$$\|\delta\mathbf{v}(\mathbf{r})\|_{2q} = O(|\mathbf{r}|^s) \implies \|\delta p(\mathbf{r})\|_q = O(|\mathbf{r}|^{2s})$$

i.e. if $\mathbf{v}$ is Besov regular of order $(2q)$ with exponent s, then p is Besov regular of order q with exponent ($2s$). To show this, we need to use an equivalent formulation of Besov regularity in terms of band-pass filtered fields, a so-called "Paley-Littlewood criterion". It is known that for $p \geq 1$ and $0 < s < 1$ that

$$\|\delta f(\mathbf{r})\|_p = O(|\mathbf{r}|^s), \ \forall |\mathbf{r}| \leq r_0 \Longleftrightarrow \|f^{[n]}\|_p = O(2^{-ns}), \ \forall n \geq n_0$$

where

$$f^{[n]} = \bar{f}_{\ell_n} - \bar{f}_{\ell_{n-1}}, \ \ell_n = 2^{-n}\ell_0$$

is the band-pass filtered function. Notice that we have already proved the $\Longrightarrow$ direction, since $f^{[n]} = f'_{\ell_{n-1}} - f'_{\ell_n}$! For a discussion of Besov regularity from this point of view, see

M. Frazier, B. Jawerth & G. Weiss, Littlewood-Paley Theory and the Study of Function Spaces (Amer. Math. Soc., Providence, RI, 1991), Chapter 5.

We shall use this result to establish first the Besov regularity of

$$\sigma \equiv -\triangle p.$$

We have seen that

$$\begin{aligned} \bar{\sigma}_\ell &= \frac{1}{\ell^2}\int d^d r \ (\partial_i\partial_j G)_\ell(\mathbf{r})\delta v_i(\mathbf{r})\delta v_j(\mathbf{r}) \\ \Longrightarrow \ \|\bar{\sigma}_\ell\|_q &\leq \frac{1}{\ell^2}\int d^d r \ |(\partial_i\partial_j G)_\ell(\mathbf{r})| \, \|\delta \mathbf{v}(\mathbf{r})\|_{2q}^2 \text{ by Hölder inequality} \end{aligned}$$

Then, if $\|\delta\mathbf{v}(\mathbf{r})\|_{2q} = O(|\mathbf{r}|^s)$, it easily follows that

$$\|\bar{\sigma}_\ell\|_q = O(\ell^{2s-2})$$

But then

$$\|\sigma^{[n]}\|_q = \|\bar{\sigma}_{\ell_n} - \bar{\sigma}_{\ell_{n-1}}\|_q \leq \|\bar{\sigma}_{\ell_n}\|_q + \|\bar{\sigma}_{\ell_{n-1}}\|_q = O(\ell_n^{2s-2})$$

By the Paley-Littlewood criterion, this implies that σ is Besov regular of order q with exponent $2s - 2$. But now notice that

$$-\triangle p = \sigma$$

where σ has the stated Besov regularity. It then follows by elliptic regularization that the solution p is Besov regular of order q with exponent

$$(2s-2)+2=2s,$$

i.e. inverting $-\triangle$ adds two orders of derivatives! E.g. see H. Triebel, Theory of Function Spaces (Birkhäuser Verlag, Basel, 1983), Section 4.2.4.

If we define pressure structure functions

$$\begin{aligned} S_q^{(p)}(\mathbf{r}) &= \langle |\delta p(\mathbf{r})|^q \rangle \\ &= \|\delta p(\mathbf{r})\|_q^q \end{aligned}$$

and corresponding scaling exponents

$$S_q^{(p)}(\mathbf{r}) \sim u_{rms}^{2q} \left(\frac{|\mathbf{r}|}{L}\right)^{\zeta_q^{(p)}}$$

analogous to those defined before for the velocity, $\zeta_q^{(u)}$, then the previous result implies that

$$\zeta_q^{(p)} \geq \zeta_{2q}^{(u)}.$$

In K41 theory one would expect that

$$\zeta_q^{(p)} = \zeta_{2q}^{(u)} = \tfrac{2}{3}q.$$

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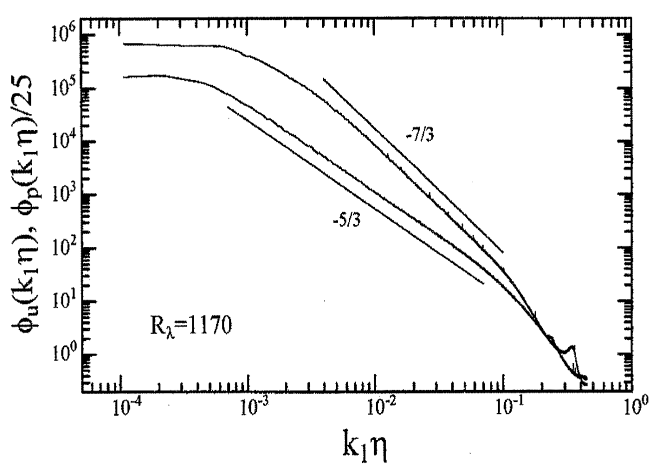


Experimental Data on the Energy & Pressure Spectra from Tsuji & Ishihara (2003). The data were taken on the center line of a free jet with $200 \leq Re_\lambda \leq 1200$. The measurement of pressure fluctuation in the flow field was accomplished with a small piezoresistive transducer (low-k) and a quarter-inch condenser microphone (high-k).

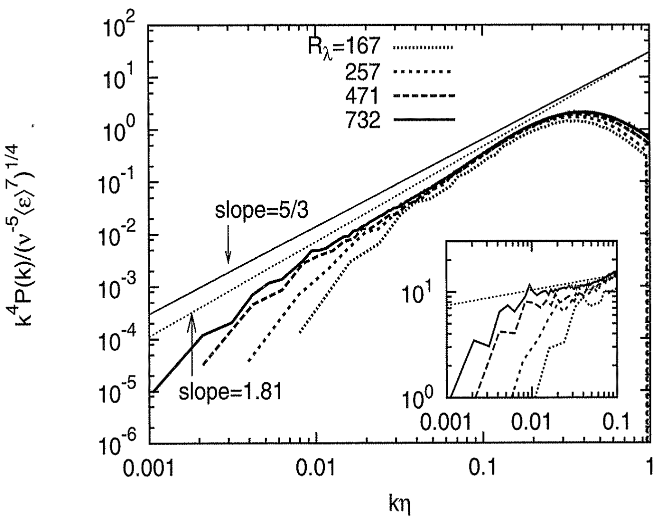
The K41 prediction was by

A. M. Obukhov, "Pressure fluctuations in a turbulent flow," Dokl. Akad. Nauk, SSSR, **66** (1), 17-20(1949).

A. M. Yaglom, "Acceleration field in a turbulent flow," Dokl. Akad. Nauk, SSSR, **67** (5), 795-798(1949).

G. K. Batchelor, "Pressure fluctuations in isotropic turbulence," Prov. Camb. Phil. Soc. **47** (2) 359-374(1951).

Numerical simulations give some evidence of an intermittency correction:



Plots of Spectrum of the Pressure Laplacian $\triangle p$.

More precisely, normalized spectra $k^4P(k)/(\nu^{-5}\langle\varepsilon\rangle^4)^{1/4}$ versus $k\eta$ in 256^3,512^3, 1024^3,2048^3 DNS. The inset shows the compensated spectra multiplied by $(k\eta)^{-5/3}$, for $0.001 < k\eta < 0.1$; the straight line shows the slope $\propto k^{1.81-5/3}$. From T. Ishihara et al., J. Phys. Soc. Japan, vol. 72, pp. 983-986 (2003).

We now consider the last two terms in the large-scale equation, namely, $\partial_t\bar{\mathbf{v}}_\ell$ and $(\bar{\mathbf{v}}_\ell\cdot\boldsymbol{\nabla})\bar{\mathbf{v}}_\ell$. Note that

$$\bar{\mathbf{v}}_\ell = \mathbf{v} - \mathbf{v}'_\ell = \mathbf{v} + O(\delta v(\ell))$$

and $\delta v(\ell) \to 0$ for $\ell \ll L$. For example, in K41 theory, $\delta v(\ell) \sim (\varepsilon\ell)^{1/3}$. Thus,

$$\bar{\mathbf{v}}_\ell \approx \mathbf{v},\ \ \ell \ll L$$

and

$$\bar{\mathbf{v}}_\ell \cdot \boldsymbol{\nabla}\bar{\mathbf{v}}_\ell = O(u_{max} \cdot \frac{\delta v(\ell)}{\ell}).$$

This is much larger than any of the terms in the equation that we have previously examined, except possibly $\bar{\mathbf{f}}_\ell$. Hence, if $\bar{\mathbf{f}}_\ell = 0$ (as in most natural flows!), then the only term that can balance it is $\partial_t\bar{\mathbf{v}}_\ell$, so that $\partial_t\bar{\mathbf{v}}_\ell \approx -(\bar{\mathbf{v}}_\ell \cdot \boldsymbol{\nabla})\bar{\mathbf{v}}_\ell$ and

$$\partial_t\bar{\mathbf{v}}_\ell = O(u_{max} \cdot \frac{\delta v(\ell)}{\ell}).$$

The combined term, or (large-scale) Lagrangian time-derivative

$$\bar{D}_t\bar{\mathbf{v}}_\ell = \partial_t\bar{\mathbf{v}}_\ell + (\bar{\mathbf{v}}_\ell \cdot \boldsymbol{\nabla})\bar{\mathbf{v}}_\ell$$

must have the magnitude of the remaining terms

$$\bar{D}_t\bar{\mathbf{v}}_\ell = O(\frac{\delta v^2(\ell)}{\ell}).$$

This implies that the Lagrangian time-scale at length ℓ must be of the order of $t_\ell = \ell/\delta v(\ell)$, the so-called local eddy-turnover time.

A summary of all these estimates is

$$\partial_t\bar{\mathbf{v}}_\ell = O(u_{max}\frac{\delta v(\ell)}{\ell})$$
$$(\bar{\mathbf{v}}_\ell \cdot \boldsymbol{\nabla})\bar{\mathbf{v}}_\ell = O(u_{max}\frac{\delta v(\ell)}{\ell})$$
$$\bar{D}_t\bar{\mathbf{v}}_\ell = O(\frac{\delta v^2(\ell)}{\ell})$$
$$\mathbf{f}^s_\ell = -\boldsymbol{\nabla} \cdot \boldsymbol{\tau}_\ell = O(\frac{\delta v^2(\ell)}{\ell})$$
$$\boldsymbol{\nabla}\bar{p}_\ell = O(\frac{\delta p(\ell)}{\ell}) = O(\frac{\delta v^2(\ell)}{\ell})$$
$$\nu\triangle\bar{\mathbf{v}}_\ell = O(\nu\frac{\delta v(\ell)}{\ell^2}) = O(\frac{\delta v^2(\ell)}{\ell} \cdot Re_\ell^{-1})$$
$$\bar{\mathbf{f}}_\ell = O(f_{max}) \quad \text{[if present]}$$

These may be compared with the estimates developed by Tennekes & Lumley, Ch.2. Note that their ℓ always corresponds to the integral length-scale or L, while we have derived estimates for all $\ell \leq L$. Also,

$$\ell \approx L \Longrightarrow \delta v \approx u_{max} \approx u_{rms}$$

Estimation of Terms in Energy Balances

We now estimate the various terms in both the large-scale and small-scale energy balances.

(i) Large-scale Energy Balance

See T & L, Section 3.1

We may write the large-scale balance as

$$\bar{D}_t \bar{e}_\ell + (\bar{\mathbf{v}}_\ell \cdot \boldsymbol{\nabla})\bar{p}_\ell + \boldsymbol{\nabla} \cdot (\boldsymbol{\tau}_\ell \cdot \bar{\mathbf{v}}_\ell) - \nu \triangle \bar{e}_\ell = -\Pi_\ell - \nu |\boldsymbol{\nabla} \bar{\mathbf{v}}_\ell|^2 + \bar{\mathbf{v}}_\ell \cdot \bar{\mathbf{f}}_\ell$$

We have already estimated all of the terms on the righthand side, except the forcing term

$$\bar{\mathbf{v}} \cdot \bar{\mathbf{f}}_\ell = O(u_{max} f_{max})$$

The terms on the lefthand side can be estimated from our previous work. It is not hard to check that the largest contributions to the four lefthand terms (from the left to right) are from

$$\bar{\mathbf{v}}_\ell \cdot (\bar{D}_t \bar{\mathbf{v}}_\ell), \quad \bar{\mathbf{v}}_\ell \cdot (\boldsymbol{\nabla} \bar{p}_\ell), \quad (\boldsymbol{\nabla} \cdot \boldsymbol{\tau}_\ell) \cdot \bar{\mathbf{v}}_\ell, \quad -(\nu \triangle \bar{\mathbf{v}}_\ell) \cdot \bar{\mathbf{v}}_\ell,$$

respectively. Of course, these four contributions sum up to give $\bar{\mathbf{v}}_\ell \cdot \bar{\mathbf{f}}_\ell$ exactly, since $\bar{D}_t \bar{\mathbf{v}}_\ell + \boldsymbol{\nabla} \bar{p}_\ell + \boldsymbol{\nabla} \cdot \boldsymbol{\tau}_\ell - \nu \triangle \bar{\mathbf{v}}_\ell = \bar{\mathbf{f}}_\ell$. We see then that the following estimates hold:

$$\bar{D}_t \bar{e}_\ell = O(u_{max} \tfrac{(\delta v)^2}{\ell})$$
$$(\bar{\mathbf{v}}_\ell \cdot \boldsymbol{\nabla})\bar{p}_\ell = O(u_{max} \tfrac{(\delta v)^2}{\ell})$$
$$\boldsymbol{\nabla} \cdot (\boldsymbol{\tau}_\ell \cdot \bar{\mathbf{v}}_\ell) = O(u_{max} \tfrac{(\delta v)^2}{\ell})$$
$$\nu \triangle \bar{e}_\ell = O(u_{max} \tfrac{\nu \delta v}{\ell^2}) = O(u_{max} \tfrac{(\delta v)^2}{\ell} \cdot Re_\ell^{-1})$$
$$\Pi_\ell = O(\tfrac{(\delta v)^3}{\ell})$$
$$\nu |\boldsymbol{\nabla} \bar{\mathbf{v}}_\ell|^2 = O(\nu \tfrac{(\delta v)^2}{\ell^2}) = O(\tfrac{(\delta v)^3}{\ell} \cdot Re_\ell^{-1})$$
$$\bar{\mathbf{v}}_\ell \cdot \bar{\mathbf{f}}_\ell = O(u_{max} f_{max}).$$

Integrating over space and assuming that there is no transport of energy across the boundary, the space transport terms must all vanish (including $(\bar{\mathbf{v}}_\ell \cdot \boldsymbol{\nabla})\bar{e}_\ell$) so that the dominant terms must be in the set

$$\partial_t \bar{e}_\ell \cong \partial_t e, \ \Pi_\ell, \ \bar{\mathbf{v}}_\ell \cdot \bar{\mathbf{f}}_\ell \cong \mathbf{v} \cdot \mathbf{f}.$$

Which terms give the dominant balance depends upon the precise situation:

In steady-state forced turbulence, taking time-averages

$$\langle \partial_t e \rangle_{\rm time} = 0, \ \langle \Pi_\ell \rangle_{\rm time} \cong \langle \mathbf{v} \cdot \mathbf{f} \rangle_{\rm time} \ \text{(independent of } \ell!)$$

In homogenous decaying turbulence, taking global space average

$$-\langle \partial_t e \rangle_{\rm space} \ \cong \ \langle \Pi_\ell \rangle_{\rm space} \quad \text{(again independent of } \ell!)$$

In other cases, the balance may be different. For example, in steady-state pipe flow, there is space transport of energy into each cross-section of the pipe, by the pressure head. Taking an average over time and also over space from a cross-section at point 1 to a cross-section at another point 2 downstream gives

$$\nu \langle |\boldsymbol{\nabla} \bar{\mathbf{v}}_\ell|^2 \rangle_{\rm space-time} + \langle \Pi_\ell \rangle_{\rm space-time}$$

$$\cong [\langle pu \rangle_{\text{cross-section 1 \& time}} - \langle pu \rangle_{\text{cross-section 2 \& time}}] / L_{12}$$

where L_{12} is the axial distance between points 1 and 2. Again one finds a range of length-scales $L \gg \ell \gg \eta$ where the energy flux is constant (note that L depends on the distance to the pipe wall!) Of course, for $\ell \approx \eta$ the term $\nu |\boldsymbol{\nabla} \bar{\mathbf{v}}_\ell|^2$ which we have neglected becomes significant and the mean flux is no longer constant. In the case of pipe flow, there is always some region close the side wall where this term cannot be neglected. E.g. see Figure 5.5, in T & L, Section 5.2.

(ii) Small-scale energy balance & stress production

See T & L, Section 3.2

We now estimate the terms in the small-scale energy balance, which may be written as

$$\begin{aligned} \bar{D}_t k_\ell \quad &+ \quad \partial_i \tau_\ell(p, v_i) + \frac{1}{2} \partial_i \tau_\ell(v_j, v_j, v_i) \\ &= \quad +\Pi_\ell - \underbrace{\nu \tau_\ell(v_{i,j}, v_{i,j})}_{\varepsilon'_\ell} + \underbrace{\tau_\ell(v_i, f_i)}_{Q'_\ell} \end{aligned}$$

In fact, we shall do something more general and estimate the terms in the small-scale stress production equation

$$\begin{aligned} \bar{D}_t \tau_\ell(v_i, v_j) \quad &+ \quad [\partial_i \tau_\ell(p, v_j) + \partial_j \tau_\ell(p, v_i)] + \partial_k \tau_\ell(v_i, v_j, v_k) \\ &= \quad -[\bar{v}_{i,k} \tau_\ell(v_k, v_j) + \tau_\ell(v_i, v_k) \bar{v}_{j,k}] \ \rightsquigarrow \text{stress production by large-scale strain} \\ &\quad\ +2\tau_\ell(p, S_{ij}) \quad \rightsquigarrow \text{pressure-strain correlation} \end{aligned}$$

$$-2\nu\tau_\ell(v_{i,k},v_{j,k}) \quad \rightsquigarrow \text{viscous destruction of stress}$$
$$+[\tau_\ell(v_i,f_j)+\tau_\ell(v_j,f_i)] \quad \rightsquigarrow \text{stress production by large-scale force}$$

Some of the terms are easy to estimate by our previous techniques:

$$\begin{aligned}
[\partial_i\tau_\ell(p,v_j)+\partial_j\tau_\ell(p,v_i)] &= O(\frac{\delta p\delta v}{\ell}) = O(\frac{(\delta v)^3}{\ell}) \\
[\bar{v}_{i,k}\tau_\ell(v_k,v_j)+\bar{v}_{j,k}\tau_\ell(v_k,v_i)] &= O(\frac{\delta v}{\ell}\cdot(\delta v)^2) = O(\frac{(\delta v)^3}{\ell}) \\
[\tau_\ell(v_i,f_j)+\tau_\ell(v_j,f_i)] &= O(\delta v\,\delta f)
\end{aligned}$$

If $\mathbf{f}$ is a smooth (large-scale) force, then $\delta f(\ell) = O(\ell\|\nabla\mathbf{f}\|_\infty)$. This term is negligible compared to the others if ℓ is small enough that

$$t_\ell = \frac{\ell}{\delta v(\ell)} \ll \sqrt{\|\nabla\mathbf{f}\|_\infty} = T_f = \text{time-scale of large-scale force}$$

The next term that we consider is the space-transport of stress by triple-correlation $\partial_k\tau_\ell(v_i,v_j,v_k)$. Here it is useful to note the identity

$$\begin{aligned}
\tau_\ell(v_i,v_j,v_k) &= \langle\delta v_i\delta v_j\delta v_k\rangle_\ell - \langle\delta v_i\delta v_j\rangle_\ell\langle\delta v_k\rangle_\ell \\
&\quad -\langle\delta v_i\delta v_k\rangle_\ell\langle\delta v_j\rangle_\ell - \langle\delta v_j\delta v_k\rangle_\ell\langle\delta v_i\rangle_\ell \\
&\quad +2\langle\delta v_i\rangle_\ell\langle\delta v_j\rangle_\ell\langle\delta v_k\rangle_\ell
\end{aligned}$$

where $\langle . \rangle_\ell$ is the average over the seperation vector $\mathbf{r}$ with respect to the filter kernel $G_\ell(\mathbf{r})$. This identity can be verified by a direct computation. However, it is useful to give a more general derivation. Suppose that $\{f_i|i\in I\}$ are any set of fields. Note that

$$\begin{aligned}
\overline{(f_{i_1}\dots f_{i_p})}_\ell(\mathbf{x}) &= \int d^dr\; G_\ell(\mathbf{r})f_{i_1}(\mathbf{x}+\mathbf{r})\dots f_{i_p}(\mathbf{x}+\mathbf{r}) \\
&= \langle(\sigma f_{i_1})\dots(\sigma f_{i_p})\rangle_\ell(\mathbf{x})
\end{aligned}$$

where

$$(\sigma f_i)(\mathbf{x}) = f_i(\mathbf{x}+\mathbf{r})$$

is the shift operator. We thus see that $\overline{f_{i_1}\dots f_{i_p}}$ is a correlation function of the "random variables" σf_{i_1},...,σf_{i_p}. Likewise, the generalized central moments defined by Germano are the connected correlation functions of σf_{i_1},...,σf_{i_p}:

$$\tau_\ell(f_{i_1},\dots,f_{i_p}) = \langle(\sigma f_{i_1})\dots(\sigma f_{i_p})\rangle_\ell^c \ .$$

There is a powerful method in statistical mechanics for computing these objects using so-called generating functions. The generating function for the correlation function is

$$Z_\ell^\sigma(\boldsymbol{\alpha}) \ = \ \langle \exp(\sum_{i\in I} \alpha_i\, \sigma f_i)\rangle_\ell.$$

It is easy to check that

$$\overline{(f_{i_1}\dots f_{i_p})_\ell} = \frac{\partial^p}{\partial\alpha_{i_1}\dots\partial\alpha_{i_p}} Z_\ell^\sigma(\boldsymbol{\alpha})|_{\boldsymbol{\alpha}=\mathbf{0}}$$

The corresponding cumulants or connected correlation functions are generated by the function

$$W_\ell^\sigma(\boldsymbol{\alpha}) = \ln Z_\ell^\sigma(\boldsymbol{\alpha}),$$

i.e.

$$\tau_\ell(f_{i_1}\dots f_{i_p}) = \tfrac{\partial^p}{\partial\alpha_{i_1}\dots\partial\alpha_{i_p}} W_\ell^\sigma(\boldsymbol{\alpha})|_{\boldsymbol{\alpha}=\mathbf{0}}.$$

This is the so-called linked cluster-theorem. See

K. Huang, Statistical Mechanics, 2nd Ed. John Wiley & Sons, NY, 1987, Section 10.1.

On the other hand, rather than the shift fields, one can consider instead the increment fields

$$\delta_{\mathbf{r}} f_i(\mathbf{x}) = \sigma_{\mathbf{r}} f_i(\mathbf{x}) - f_i(\mathbf{x}).$$

The correlation functions of the increments $\langle \delta f_{i_1}\dots\delta f_{i_p}\rangle_\ell$ are generated by the function

$$Z_\ell^\delta(\boldsymbol{\alpha}) \ = \ \langle \exp(\sum_{i\in I} \alpha_i\, \delta f_i)\rangle_\ell,$$

and the connected correlation functions by the function

$$W_\ell^\delta(\boldsymbol{\alpha}) = \ln Z_\ell^\delta(\boldsymbol{\alpha}),$$

i.e.

$$\langle(\delta f_{i_1})\dots(\delta f_{i_p})\rangle_\ell^c = \tfrac{\partial^p}{\partial\alpha_{i_1}\dots\partial\alpha_{i_p}} W_\ell^\delta(\boldsymbol{\alpha})|_{\boldsymbol{\alpha}=\mathbf{0}},$$

again by the linked-cluster theorem. Now comes the key observation: since $f_i(\mathbf{x})$ does not depend on $\mathbf{r}$, it can be taken outside the average $\langle . \rangle_\ell$. Thus, using $\delta f_i = \sigma f_i - f_i$,

$$\begin{aligned} Z_\ell^\delta(\boldsymbol{\alpha}) \quad &= \quad \langle \exp(\sum_{i\in I} \alpha_i\, \delta f_i)\rangle_\ell, \\ &= \quad \langle \exp(\sum_{i\in I} \alpha_i\, \sigma f_i)\rangle_\ell \exp(-\sum_{i\in I} \alpha_i f_i) \end{aligned}$$

$$= \quad Z^{\sigma}_{\ell}(\boldsymbol{\alpha}) \exp(-\sum_{i \in I} \alpha_i f_i)$$

Taking the logarithm of both sides then gives

$$W^{\delta}_{\ell}(\boldsymbol{\alpha}) = W^{\sigma}_{\ell}(\boldsymbol{\alpha}) - \sum_{i \in I} \alpha_i f_i$$

Since they differ only by a term linear in $\boldsymbol{\alpha}$, we can draw our main conclusion:

<u>Proposition</u>: The connected correlation functions of δf_i and σf_i are related for $p = 1$ by

$$\langle \delta f_i \rangle_{\ell} \;=\; \langle \sigma f_i \rangle_{\ell} - f_i$$

and for $p > 1$ are equal

$$\langle \delta f_{i_1} \ldots \delta f_{i_p} \rangle^c_{\ell} \;=\; \langle \sigma f_{i_1} \ldots \sigma f_{i_p} \rangle^c$$

In terms of the quantities defined by Germano (1992), this means that

$$(f_i)'_{\ell} = f_i - \overline{(f_i)_{\ell}} = -\langle \delta f_i \rangle_{\ell}$$

and, for $p > 1$,

$$\tau_{\ell}(f_{i_1}, \ldots, f_{i_p}) = \langle \delta f_{i_1} \ldots \delta f_{i_p} \rangle^c_{\ell}.$$

For example, for $p = 2$

$$\tau_{\ell}(f_i, f_j) = \langle \delta f_i \delta f_j \rangle_{\ell} - \langle \delta f_i \rangle_{\ell} \langle \delta f_j \rangle_{\ell}$$

and for $p = 3$

$$\begin{aligned} \tau_{\ell}(f_i, f_j, f_k) \quad = \quad & \langle \delta f_i \delta f_j \delta f_k \rangle_{\ell} - \langle \delta f_i \delta f_j \rangle_{\ell} \langle \delta f_k \rangle_{\ell} \\ & - \langle \delta f_i \delta f_k \rangle_{\ell} \langle \delta f_j \rangle_{\ell} - \langle \delta f_j \delta f_k \rangle_{\ell} \langle \delta f_i \rangle_{\ell} \\ & + 2 \langle \delta f_i \rangle_{\ell} \langle \delta f_j \rangle_{\ell} \langle \delta f_k \rangle_{\ell}, \qquad \text{etc.!!!} \end{aligned}$$

After this somewhat lengthy interlude, we conclude that

$$\tau(v_i, v_j, v_k) = O((\delta v)^3).$$

Furthermore, using the same "shift trick" as for the 2nd-order term

$$\partial_m \tau_\ell(v_i, v_j, v_k) = -\frac{1}{\ell}\Bigg\{\int d^d r (\partial_m G)_\ell(\mathbf{r})\delta v_i(\mathbf{r})\delta v_j(\mathbf{r})\delta v_k(\mathbf{r})$$
$$-\Bigg[\int d^d r (\partial_m G)_\ell(\mathbf{r})\delta v_i(\mathbf{r})\delta v_j(\mathbf{r})\int d^d r'\ G_\ell(\mathbf{r}')\delta v_k(\mathbf{r}')$$
$$+\int d^d r G_\ell(\mathbf{r})\delta v_i(\mathbf{r})\delta v_j(\mathbf{r})\int d^d r' (\partial_m G)_\ell(\mathbf{r}')\delta v_k(\mathbf{r}')$$
$$+\text{ cyclic permutations among } i,j,k\Bigg]$$
$$-2\Bigg[\int d^d r (\partial_m G)_\ell(\mathbf{r})\delta v_i(\mathbf{r})\int d^d r' G_\ell(\mathbf{r}')\delta v_j(\mathbf{r}')\int d^d r'' G_\ell(\mathbf{r}'')\delta v_k(\mathbf{r}'')$$
$$+\text{ cyclic permutations among } i,j,k\Bigg]\Bigg\}$$

From this one obtains

$$\partial_k \tau(v_i, v_j, v_k) = O(\tfrac{(\delta v)^3}{\ell}) \ \text{ ...finally!}$$

The next term that we examine in the stress balance is the viscous destruction term:

$$\varepsilon'_{\ell ij} = 2\nu\tau_\ell(v_{i,k}, v_{j,k})$$

This satisfies matrix positivity:

$$\boldsymbol{\varepsilon}'_\ell \geq 0$$

The Cauchy-Schwarz inequality also implies that

$$\begin{aligned} \varepsilon'_{\ell ij} &\leq 2\nu\sqrt{\tau_\ell(v_{i,k}, v_{i,k})\tau_\ell(v_{j,k}, v_{j,k}} \\ &= \sqrt{\varepsilon'_{\ell ii}\varepsilon'_{\ell jj}} \end{aligned}$$

Note that $\mathrm{Tr}(\boldsymbol{\varepsilon}'_\ell) = \sum_i \varepsilon'_{\ell ii} = \varepsilon'_\ell$ is the subscale dissipation. Thus, in particular,

$$\varepsilon'_{\ell ij} \leq \varepsilon'_\ell \ \text{ for all } \ i, j.$$

Aside from these rigorous bounds, it is difficult to develop exact estimates for $\boldsymbol{\varepsilon}'_\ell$, because $\boldsymbol{\nabla}\mathbf{v}$ is a dissipation-range variable which does not remain Hölder continuous/Besov regular (or even L_p!) in the limit as $\nu \to 0$. We can get some additional insight, however, by decomposing the viscous destruction into two parts:

$$\varepsilon'_{\ell,ij} = 2\nu\overline{(v_{i,k}v_{j,k})_\ell} - 2\nu(\bar{v}_{i,k})_\ell(\bar{v}_{j,k})_\ell$$

Since $\boldsymbol{\nabla}\bar{\mathbf{v}}_\ell = O(\delta v(\ell)/\ell)$, we can see that the second term can be estimated as

$$2\nu\boldsymbol{\nabla}\bar{\mathbf{v}}_\ell(\boldsymbol{\nabla}\bar{\mathbf{v}}_\ell)^\top = O(\nu\frac{\delta v^2(\ell)}{\ell^2}) = O(\frac{\delta v^3(\ell)}{\ell}\cdot Re_\ell^{-1})$$

with $Re_\ell = \frac{\delta v(\ell)\ell}{\nu}$. This is small compared with the other terms in the stress production equation. However, the first term is expected to be bigger and, in fact, it is expected that

$$2\nu\overline{(v_{i,k}v_{j,k})_\ell} \sim (const.)\frac{\delta v^3(\ell)}{\ell}$$

For the trace, i.e. for the viscous dissipation, this is a famous conjecture of A.N.Kolmogorov (1962) the refined similarity hypothesis (RSH)

$$\overline{\varepsilon}_\ell(\mathbf{x}) = \overline{(2\nu|\boldsymbol{\nabla}\mathbf{v}|^2)_\ell} = \int d\mathbf{r}\ G_\ell(\mathbf{r})\varepsilon(\mathbf{x}+\mathbf{r}) \sim (const.)\frac{\delta v^3(\ell)}{\ell}$$

This estimate is certainly consistent with K41, since then $|\boldsymbol{\nabla}\mathbf{v}|^2 \sim (\overline{\langle\varepsilon\rangle}/\nu)$ everywhere, but also $\delta v(\ell) \sim (\overline{\langle\varepsilon\rangle}\ell)^{1/3}$, so that both sides are proportional to $\overline{\langle\varepsilon\rangle}$. But RSH is also consistent with intermittancy, which we discuss later. It is reasonable to extend these ideas also to the off-diagonal terms. At least, it is reasonable to guess that there is an upper bound:

$$\varepsilon'_\ell = O(\frac{\delta v^3(\ell)}{\ell})$$

The last — and most difficult! — term to estimate is the pressure-strain correlation

$$\Phi_{\ell,ij} = 2\tau_\ell(p, S_{ij})$$

This term is especially tricky, because it is a mixed quantity, with p inertial-range and S_{ij} dissipation range. Thus,

$$\begin{aligned}\boldsymbol{\Phi}_\ell &= O(\delta p(\ell)|\mathbf{S}|)\\ &= O(\delta v^2(\ell)\cdot(\frac{\varepsilon}{\nu})^{1/2})\end{aligned}$$

However, this upper bound is expected to be a big overestimate. Because p and S_{ij} "live" on different length-scales, they are presumably very poorly correlated in space. Thus, the local average over the region of radius $\sim \ell$ that defines $\boldsymbol{\Phi}_\ell$ should have substantial cancellations! Following T & L, section 3.2, we may estimate the correlation coefficient between these term by the ratio of time-scales

$$\rho(\delta p(\ell), S) \sim \frac{t_\eta}{t_\ell} \sim \frac{\delta v(\ell)/\ell}{(\varepsilon/\nu)^{1/2}} \ll 1.$$

We can therefore expect that, in fact,

$$\boldsymbol{\Phi}_\ell = O(\delta v^2(\ell)\cdot(\frac{\varepsilon}{\nu})^{1/2}\cdot\rho(\delta p(\ell), S)) = O(\frac{\delta v^3(\ell)}{\ell})!$$

Lastly, we note that $\bar{D}_t \tau_\ell(v_i, v_j) = O(\frac{\delta v^3(\ell)}{\ell})$, since it is equal to the sum of all the other terms and these are, at most, of that magnitude. This is a reasonable estimate of that term since $\tau_{ij} = O(\delta v^2(\ell))$ and we can expect that $\bar{D}_{\ell,t} = O(\delta v(\ell)/\ell)$. We thus obtain, finally,

$$\begin{aligned}
\bar{D}_t \tau_\ell(v_i, v_j) &= O(\frac{\delta v^3(\ell)}{\ell})^\star \\
\partial_i \tau_\ell(p, v_j) + \partial_j \tau_\ell(p, v_i) &= O(\frac{\delta v^3(\ell)}{\ell}) \\
\partial_k \tau_\ell(v_i, v_j, v_k) &= O(\frac{\delta v^3(\ell)}{\ell}) \\
\bar{v}_{i,k} \tau_\ell(v_k, v_j) + \bar{v}_{j,k} \tau_\ell(v_{i,k}) &= O(\frac{\delta v^3(\ell)}{\ell}) \\
2\tau_\ell(p, S_{ij}) &= O(\frac{\delta v^3(\ell)}{\ell})^\star \\
2\nu \tau_\ell(v_{i,k}, v_{j,k}) &= O(\frac{\delta v^3(\ell)}{\ell})^\star \\
\tau_\ell(v_i, f_j) + \tau_\ell(v_j, f_i) &= O(\delta v(\ell)\, \delta f(\ell))
\end{aligned}$$

[1]Note that the terms marked with ($\star$) are not completely rigorous upper estimates but only heuristic.