II Effective "Large-Scale" Equations and the Turbulent Energy Cascade

(A) Coarse-Graining/Filtering/Mollifying

We have seen that the non-vanishing of turbulent energy dissipation as $Re \to \infty$ requires that velocity-gradients $|\boldsymbol{\nabla}\mathbf{u}| \to \infty$ in this limit. But, in that case, the usual formulation of the fluid equations as PDE's with smooth solutions no longer makes sense! To obtain a dynamical description, we must *regularize* those equations. We shall use an approach based on coarse-graining/filtering/mollifying with a smooth kernel G that satisfies

$$\begin{aligned} G(\mathbf{r}) &\geq 0 \\ G(\mathbf{r}) &\to 0 \text{ rapidly for } \ |\mathbf{r}| \to \infty \\ \int d^d r\ G(\mathbf{r}) &= 1 \end{aligned}$$

It is also understood that G is centered at $\mathbf{r} = \mathbf{0}$:

$$\int d^d r\ \mathbf{r}\ G(\mathbf{r}) = \mathbf{0},$$

and that $\int d^d r |\mathbf{r}|^2 G(\mathbf{r}) \approx 1$. Other specific requirements shall be introduced as needed. Set

$$G_\ell(\mathbf{r}) \equiv \ell^{-d} G(\mathbf{r}/\ell)$$

so that all of the above properties hold, except that now $\int d^d r |\mathbf{r}|^2 G_\ell(\mathbf{r}) \approx \ell^2$.

Using this kernel, now define a coarse-grained velocity at length-scale ℓ by

$$\bar{\mathbf{v}}_\ell(\mathbf{x}) = \int d^d r\, G_\ell(\mathbf{r}) \mathbf{v}(\mathbf{x} + \mathbf{r})$$

This represents the average velocity of a fluid "parcel" of size ℓ at position $\mathbf{x}$. It can also be called a low-pass filtered velocity, containing only length-scales $> \ell$, or a mollified (i.e. smoothed) velocity. The corresponding small-scale / high-pass filtered velocity is given by

$$\mathbf{v}'_\ell(\mathbf{x}) \equiv \mathbf{v}(\mathbf{x}) - \bar{\mathbf{v}}_\ell(\mathbf{x}) = -\int d^d r\ G_\ell(\mathbf{r}) \delta\mathbf{v}(\mathbf{r}; \mathbf{x}) \quad (1)$$

where

$$\delta\mathbf{v}(\mathbf{r};\mathbf{x}) = \mathbf{v}(\mathbf{x}+\mathbf{r}) - \mathbf{v}(\mathbf{x})$$

is the velocity-increment across a separation vector $\mathbf{r}$ at point $\mathbf{x}$

Comments:

$\star$ This coarse-graining is similar to that used to derive hydrodynamics from MD. However, in that case $\ell \ll L_{\boldsymbol{\nabla}} =$ gradient length $\cong \frac{|\mathbf{v}|}{|\boldsymbol{\nabla}\mathbf{v}|}$ whereas here we have in mind $\ell \gg L_{\boldsymbol{\nabla}}$.

$\star$ In principle, only coarse-grained fields $\bar{\mathbf{v}}_\ell(\mathbf{x})$ are experimentally measurable. Every experiment has some spatial resolution ℓ, such that only averaged properties for length-scales $\geq \ell$ are obtained. The fine-grained/bare field $\mathbf{v}(\mathbf{x})$ are unobservable objects corresponding to a mathematical idealization

$$\bar{\mathbf{v}}_\ell(\mathbf{x}) \to \mathbf{v}(\mathbf{x}) \text{ as } \ell \to 0$$

This idealization is physically unachievable, in the strictest sense, since the hydrodynamic equations are not valid for $\ell \approx \lambda$, the mean-free path. In general, $\bar{\mathbf{v}}_\ell(\mathbf{x})$ is a more physical object and $\mathbf{v}(\mathbf{x})$ is an "ideal" object which is useful if $L_{\boldsymbol{\nabla}} \gg \ell \gg \lambda$.

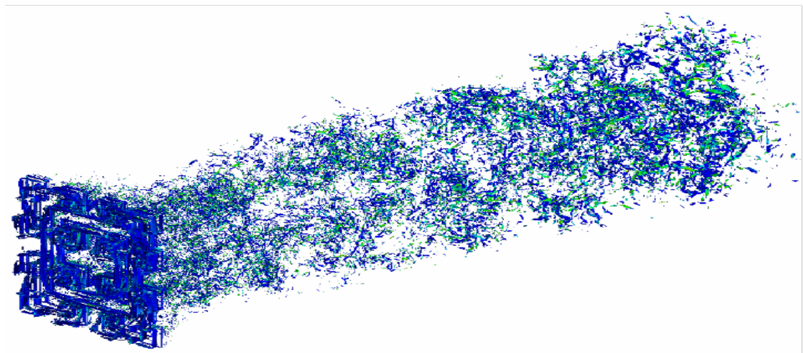
$\star$ In physics, the coarse-grained field is similar to an "effective block-spin" that appears in the method of real-space renormalization group (RG). It removes the ultraviolet divergence associated with blow-up of velocity gradients $|\boldsymbol{\nabla}\mathbf{v}| \to \infty$ as $\nu \to 0$, since necessarily

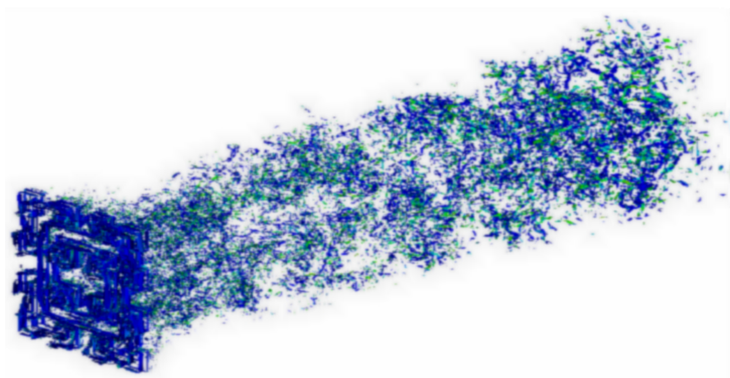
$$\boldsymbol{\nabla}\bar{\mathbf{v}}_\ell(\mathbf{x}) = -\frac{1}{\ell}\int d^d r\, (\boldsymbol{\nabla}G)_\ell(\mathbf{r})\ \mathbf{v}(\mathbf{x}+\mathbf{r})$$

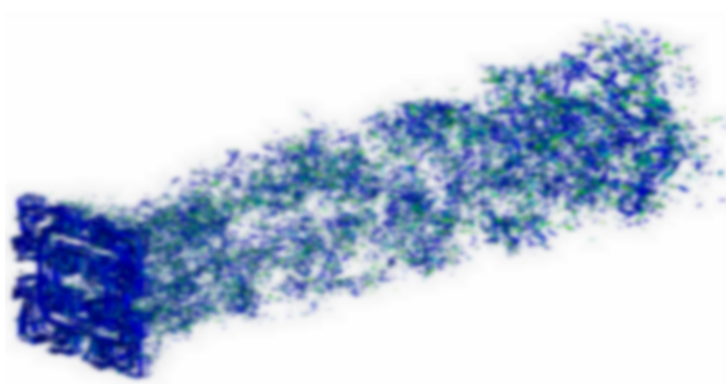
remains finite. This "regularization" introduces an arbitrary length-scale ℓ, on which no objective physical fact can depend. Note that coarse-graining is a purely passive operation—"removing one's spectacles"—which changes no physical process.

The method of filtering is also employed as part of the large-eddy simulation (LES) modelling technique for turbulent flow. Here a seminal work is:

M. Germano, "Turbulence: the filtering approach," J. Fluid Mech. **238**, 325–336 (1992).







(B) Effective Large-Scale Equations

See also T&L , Section 2.1. Starting with the incompressible Navier-Stokes equations

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = -\boldsymbol{\nabla} p + \nu\triangle\mathbf{v} + \mathbf{f}, \ \ \boldsymbol{\nabla}\cdot\mathbf{v} = 0$$

for the bare/fine-grained velocity field, we can derive an equation for $\bar{\mathbf{v}}_\ell$. Note that

$$\overline{(\boldsymbol{\nabla}\mathbf{f})}_\ell = \boldsymbol{\nabla}\bar{\mathbf{f}}_\ell,$$

i.e. space-derivatives commute with filtering. Thus,

$$\partial_t \bar{\mathbf{v}}_\ell + \boldsymbol{\nabla}\cdot\overline{(\mathbf{v}\mathbf{v})}_\ell = -\boldsymbol{\nabla}\bar{p}_\ell + \nu\triangle\bar{\mathbf{v}}_\ell + \bar{\mathbf{f}}_\ell, \ \ \boldsymbol{\nabla}\cdot\bar{\mathbf{v}}_\ell = 0$$

Define the <u>turbulent (or Reynolds) stress tensor</u>

$$\boldsymbol{\tau}_\ell = \overline{(\mathbf{v}\mathbf{v})}_\ell - \bar{\mathbf{v}}_\ell\bar{\mathbf{v}}_\ell$$

so that

$$\boxed{\partial_t \bar{\mathbf{v}}_\ell + (\bar{\mathbf{v}}_\ell\cdot\boldsymbol{\nabla})\bar{\mathbf{v}}_\ell + \boldsymbol{\nabla}\cdot\boldsymbol{\tau}_\ell = -\boldsymbol{\nabla}\bar{p}_\ell + \nu\triangle\bar{\mathbf{v}}_\ell + \bar{\mathbf{f}}_\ell}$$

This is the "effective equation" for the large-scale velocity. Note that it is <u>not closed</u>, i.e. $\boldsymbol{\tau}$ is not given (in a simple way) as a function of $\bar{\mathbf{v}}_\ell$.

We now wish to estimate the viscous term $\nu\triangle\bar{\mathbf{v}}_\ell$ as small, i.e. to show that it can be neglected relative to the other terms when ν is small or when ℓ is large. To measure "size", we need the notion of a <u>norm</u> of a function $\mathbf{f}:\mathbb{R}^d\to\mathbb{R}^d$, i.e. a mapping $\mathbf{f}\longmapsto\|\mathbf{f}\|$ such that:

(i) $\|\mathbf{f}\|\geq 0$ and $\|\mathbf{f}\| = 0$ iff $\mathbf{f} = \mathbf{0}$

(ii) $\|\alpha\mathbf{f}\| = |\alpha|\cdot\|\mathbf{f}\|$ for any real scalar α.

(iii) $\|\mathbf{f}+\mathbf{g}\|\leq\|\mathbf{f}\|+\|\mathbf{g}\|$ (triangle inequality)

Some common norms are the <u>L_p norms</u> for $p \geq 1$

$$\|\mathbf{f}\|_p \equiv \left[\frac{1}{V}\int_V d^d\mathbf{x}\ |\mathbf{f}(\mathbf{x})|^p\right]^{1/p}$$

and for $p = +\infty$

$$\|\mathbf{f}\|_\infty \equiv \sup_{x\in V} |\mathbf{f}(\mathbf{x})|$$

Note that these satisfy

$$\|\mathbf{f}\|_p \leq \|\mathbf{f}\|_{p'} \quad \text{for} \quad p' \geq p$$

and

$$\|\mathbf{f}\|_\infty = \lim_{p\to\infty} \|\mathbf{f}\|_p.$$

Some of these have simple meaning

$$\begin{aligned}
\|\mathbf{f}\|_1 &= \langle|\mathbf{f}|\rangle \quad \text{where} \quad \langle g\rangle = \frac{1}{V}\int_V d^3\mathbf{x}\ g(\mathbf{x}) \\
\|\mathbf{f}\|_2 &= \sqrt{\langle|\mathbf{f}|^2\rangle} = f_{rms} \quad \text{when } \langle\mathbf{f}\rangle = \mathbf{0} \\
\|\mathbf{f}\|_\infty &= |\mathbf{f}|_{max}
\end{aligned}$$

For more details, see A. N. Kolmogorov & S. U. Fomin, *Introductory Real Analysis*, Dover, 1975

Now we estimate

$$\nu\triangle\bar{\mathbf{v}}_\ell(\mathbf{x}) = \frac{\nu}{\ell^2}\int d^dr(\triangle G)_\ell(\mathbf{r})\mathbf{v}(\mathbf{x}+\mathbf{r}) \text{ (integration by parts!)}$$

Then[1]

$$\begin{aligned}
\|\nu\triangle\bar{\mathbf{v}}_\ell\|_p &\leq \frac{\nu}{\ell^2}\int d^dr\ |(\triangle G)_\ell(\mathbf{r})|\cdot\|\mathbf{v}(\cdot+\mathbf{r})\|_p \text{ (by triangle inequality)} \\
&= \frac{\nu}{\ell^2}(const.)\|\mathbf{v}\|_p
\end{aligned}$$

using $\|\mathbf{v}(\cdot+\mathbf{r})\|_p = \|\mathbf{v}\|_p$ and assuming that $\int d^d\rho\ |\triangle G(\boldsymbol{\rho})| < +\infty$.

[1] The "continuous" version of the triangle inequality that we use below and repeatedly in these notes is usually called the *Minkowski integral inequality*. It states that $\|\int d^dr\, F(\mathbf{r},\cdot)\|_p \leq \int d^dr\, \|F(\mathbf{r},\cdot)\|_p$. See G. H. Hardy, J. E. Littlewood, and G. Pólya, "Inequalities" (Cambridge University Press, 1952), Theorem 202, or E. Stein, "Singular Integrals and Differentiability Properties of Functions" (Princeton University Press, 1970), §A.1.

If $\|\mathbf{v}\|_p$ stays finite in the limit that $\nu \to 0$, then

$$\lim_{\nu \to 0} \|\nu \triangle \bar{\mathbf{v}}_\ell\|_p = 0 \text{ !!!}$$

Note that energy is given by

$$E(t) = \frac{1}{2}\|\mathbf{v}(t)\|_2^2$$

Since $dE/dt = -\nu\|\boldsymbol{\nabla}\mathbf{v}\|_2^2 \leq 0$, $\|\mathbf{v}(t)\|_2$ can only decrease in time. Thus, $\|\mathbf{v}(t)\|_2 \leq \|\mathbf{v}(t_0)\|_2 =$ *initial energy*. Thus for decaying turbulence and $p = 2$, it is true that $\|\mathbf{v}(t)\|_2$ must stay finite in the limit as $\nu \to 0$. Experimental evidence is that $\|\mathbf{v}(t)\|_p$ stays finite for all $p \geq 1$.

Heuristically, we may say that

$$\nu \triangle \bar{\mathbf{v}}_\ell \sim \frac{\nu U}{\ell^2}$$

We shall see later that

$$(\bar{\mathbf{v}}_\ell \cdot \boldsymbol{\nabla})\bar{\mathbf{v}}_\ell \sim \boldsymbol{\nabla} \cdot \boldsymbol{\tau}_\ell \sim \boldsymbol{\nabla}\bar{p}_\ell \sim \frac{U^2}{\ell}$$

Thus, the viscous term is smaller by a factor $\frac{\nu}{U\ell} = \frac{1}{Re_\ell}$.

One of the important terms in the large-scale equation is from the turbulent (or subscale) force

$$\mathbf{f}_\ell^s = -\boldsymbol{\nabla} \cdot \boldsymbol{\tau}_\ell$$

This can be thought of as an effective "body-force" on the large-scales produced by the eliminated subscales. Note that

Prop: The stress $(\boldsymbol{\tau}_\ell)_{ij} = \overline{(v_i v_j)_\ell} - \bar{v}_{\ell i}\bar{v}_{\ell j}$ is a symmetric, nonnegative-definite matrix at each space-time point $(\mathbf{x}, t)$.

Proof: Omit ℓ for simplicity of notation. Clearly,

$$\tau_{ij} = \overline{v_i v_j} - \bar{v}_i \bar{v}_j$$

is symmetric in i,j. Note also that

$$\begin{aligned}\tau_{ij} &= \int d^dr\ G_\ell(\mathbf{r})v_i(\mathbf{x}+\mathbf{r})v_j(\mathbf{x}+\mathbf{r}) - \bar{v}_i(\mathbf{x})\bar{v}_j(\mathbf{x}) \\ &= \int d^dr\ G_\ell(\mathbf{r})[v_i(\mathbf{x}+\mathbf{r}) - \bar{v}_i(\mathbf{x})][v_j(\mathbf{x}+\mathbf{r}) - \bar{v}_j(\mathbf{x})]\end{aligned}$$

so that

$$\sum_{ij} c_i^* c_j \tau_{ij} = \int d^dr\ G_\ell(\mathbf{r})\left|\sum_j c_j[v_j(\mathbf{x}+\mathbf{r}) - \bar{v}_j(\mathbf{x})]\right|^2 \geq 0$$

In fact, τ_{ij} is just the covariance of the random variable $v_i(\mathbf{x}+\mathbf{r})$ with $\mathbf{r}$ distributed according to the "probability density" $G_\ell(\mathbf{r})$. Thus, τ_{ij} is the velocity covariance of the fluid parcel of size ℓ at spacetime point $(\mathbf{x},t)$.

This result is due to Vreman, Geurts & Kuerten, "Realizability conditions for the turbulent stress tensor in Large Eddy Simulation," J. Fluid Mech., **278**, 351-362, 1994.

(C) Energy Balance

See T & L, Sections 3, 1-2; Frisch, Section 2.4.

We have seen that the viscous term $\nu\triangle\bar{\mathbf{v}}_\ell$ is negligible is the large-scale effective equation. Since $\|\bar{\mathbf{v}}_\ell(t)\|_2^2 \leq \|\mathbf{v}(t)\|_2^2$ by convexity, the kinetic energy decays even with "spectacles off"! How? To analyze this question, we must consider energy balance in detail.

Large-Scale Energy Balance

From the equation for $\bar{\mathbf{v}}_\ell$ it is easy to derive an evolution equation for the large-scale energy (per unit mass)

$$\bar{E}_\ell(t) = \frac{1}{2}\int d^d\mathbf{x}\ |\bar{\mathbf{v}}_\ell(\mathbf{x},t)|^2$$

and its density

$$\bar{e}_\ell(\mathbf{x},t) = \frac{1}{2}|\bar{\mathbf{v}}_\ell(\mathbf{x},t)|^2.$$

One finds by straightforward calculus that

$$\boxed{\partial_t \bar{e}_\ell + \boldsymbol{\nabla}\cdot[(\bar{e}_\ell+\bar{p}_\ell)\bar{\mathbf{v}}_\ell + \boldsymbol{\tau}_\ell\cdot\bar{\mathbf{v}}_\ell - \nu\boldsymbol{\nabla}\bar{e}_\ell] = \boldsymbol{\nabla}\bar{\mathbf{v}}_\ell : \boldsymbol{\tau}_\ell - \nu|\boldsymbol{\nabla}\bar{\mathbf{v}}_\ell|^2 + \bar{\mathbf{v}}_\ell\cdot\bar{\mathbf{f}}_\ell}$$

with

$$\begin{aligned} \bar{\mathbf{J}}_\ell^E &= (\bar{e}_\ell+\bar{p}_\ell)\bar{\mathbf{v}}_\ell + \boldsymbol{\tau}_\ell\cdot\bar{\mathbf{v}}_\ell - \nu\boldsymbol{\nabla}\bar{e}_\ell \\ &= \text{space transport (flux) of large-scale energy,} \end{aligned}$$

$$\text{(or in detail:)}\begin{cases} (\bar{e}_\ell+\bar{p}_\ell)\bar{\mathbf{v}}_\ell = \text{transport by large-scale advection} \\ -\nu\boldsymbol{\nabla}\bar{e}_\ell = \text{viscous diffusion of large-scale energy} \\ \boldsymbol{\tau}_\ell\cdot\bar{\mathbf{v}}_\ell = \text{turbulent diffusion of large-scale energy} \end{cases}$$

$$\bar{Q}_\ell = \bar{\mathbf{v}}_\ell\cdot\mathbf{f}_\ell = \text{power input from external force into large-scales (per mass)}$$

$$\bar{\varepsilon}_\ell = \nu|\boldsymbol{\nabla}\bar{\mathbf{v}}_\ell|^2 = \text{large-scale energy dissipation rate (per unit mass)}$$

$$\begin{aligned} \bar{\mathcal{E}}_\ell(t) &= \int d^d\mathbf{x}\ \bar{\varepsilon}_\ell(\mathbf{x},t) = \text{total energy dissipation (per mass) in the large-scales} \\ &= \nu\|\boldsymbol{\nabla}\bar{\mathbf{v}}_\ell(t)\|_2^2 \\ &\le \frac{\nu}{\ell^2}(const.)\|\mathbf{v}(t)\|_2^2 \text{ by Minkowski estimate of } \boldsymbol{\nabla}\bar{\mathbf{v}}_\ell = -\tfrac{1}{\ell}\textstyle\int d^dr\ (\boldsymbol{\nabla}G)_\ell(\mathbf{r})\mathbf{v}(\mathbf{x}+\mathbf{r}) \\ &\longrightarrow 0 \text{ as } \nu\to 0\ ! \end{aligned}$$

We have assumed here that $\|\mathbf{v}(t)\|_2^2$ stays bounded as $\nu\to 0$.

The important term in the large-scale balance is

$$\begin{aligned} \Pi_\ell &= -\boldsymbol{\nabla}\bar{\mathbf{v}}_\ell : \boldsymbol{\tau}_\ell \\ &= \text{deformation work of the large-scale strain against the small-scale stress} \end{aligned}$$

$$\Pi_\ell > 0 \Longrightarrow \text{large-scale sink;} \qquad \Pi_\ell < 0 \Longrightarrow \text{large-scale source}$$

Alternative forms:

$$\Pi_\ell = -\bar{\mathbf{S}}_\ell : \boldsymbol{\tau}_\ell$$

with $\bar{\mathbf{S}}_\ell = \frac{1}{2}[(\boldsymbol{\nabla}\bar{\mathbf{v}}_\ell) + (\boldsymbol{\nabla}\bar{\mathbf{v}}_\ell)^T] =$ large-scale strain (by symmetry of $\boldsymbol{\tau}_\ell$); or

$$\Pi_\ell = -\boldsymbol{\nabla}\bar{\mathbf{v}}_\ell : \mathring{\boldsymbol{\tau}}_\ell$$

with $\mathring{\boldsymbol{\tau}}_\ell = \boldsymbol{\tau}_\ell - \frac{1}{d}tr(\boldsymbol{\tau}_\ell)\mathbf{I} =$ deviatoric/traceless part of the stress (by $\boldsymbol{\nabla}\cdot\bar{\mathbf{v}}_\ell = 0$); or

$$\Pi_\ell = -\bar{\mathbf{S}}_\ell : \mathring{\boldsymbol{\tau}}_\ell \quad \text{(by both together).}$$

Where does the energy go? To the small scales! Define

$$\begin{aligned} k_\ell(\mathbf{x},t) &= \frac{1}{2}tr(\boldsymbol{\tau}_\ell) = \text{ small-scale kinetic energy (per mass)} \\ &= \frac{1}{2}\overline{(|\mathbf{v}|^2)_\ell} - \frac{1}{2}|\bar{\mathbf{v}}_\ell|^2 \geq 0 \text{ by positive-definiteness of stress } \boldsymbol{\tau}_\ell \end{aligned}$$

Note that $\bar{e}_\ell = \frac{1}{2}|\bar{\mathbf{v}}_\ell|^2$ so that

$$\bar{e}_\ell + k_\ell = \overline{\frac{1}{2}(|\mathbf{v}|^2)_\ell}$$

and thus (since $\int d^d\mathbf{x}\bar{f}(\mathbf{x}) = \int d^d\mathbf{x} f(\mathbf{x})$)

$$\begin{aligned} \int d^d\mathbf{x}\ [\bar{e}_\ell + k_\ell] &= \frac{1}{2}\int d^d\mathbf{x}\ |\mathbf{v}(\mathbf{x},t)|^2 \\ &= E(t) = \text{ total kinetic energy (per mass)} \end{aligned}$$

An evolution equation of the following form holds for k_ℓ:

$$\boxed{\partial_t k_\ell + \boldsymbol{\nabla}\cdot\mathbf{J}_\ell^{E'} = -\boldsymbol{\nabla}\bar{\mathbf{v}} : \boldsymbol{\tau}_\ell - \varepsilon'_\ell + Q'_\ell}$$

Recall that $\Pi_\ell = -\boldsymbol{\nabla}\bar{\mathbf{v}}_\ell : \boldsymbol{\tau}_\ell$. Also

$$\varepsilon'_\ell = \nu[\overline{(|\boldsymbol{\nabla}\mathbf{v}|^2)_\ell} - |\boldsymbol{\nabla}\bar{\mathbf{v}}_\ell|^2] = \text{viscous energy dissipation in the small-scales}$$

so that $\bar{\varepsilon}_\ell + \varepsilon'_\ell = \nu\overline{(|\boldsymbol{\nabla}\mathbf{v}|^2)_\ell}$ gives the total dissipation averaged over the region of radius ℓ around $\mathbf{x}$. Note that Π_ℓ appears with <u>opposite signs</u> in the equations for $\bar{e}_\ell$ and k_ℓ: it tends to act as a sink for $\bar{e}_\ell$ and a source for k_ℓ. Thus, it transfers energy from large-scales to small-scales. For this reason, Π_ℓ is often called (scale-to-scale) energy flux.

We now derive the equation for k_ℓ, following Germano (1992). In fact, we derive a more general evolution equation for $\boldsymbol{\tau}_\ell$. Starting with $\partial_t v_i = -(v_k\partial_k)v_i - \partial_i p + \nu\partial_k^2 v_i + f_i$,

$$\begin{aligned}
\partial_t\overline{v_iv_j} &= \overline{(\partial_t v_i)v_j} + \overline{v_i(\partial_t v_j)} \\
&= -\overline{v_k(\partial_k v_i)v_j} - \overline{(\partial_i p)v_j} + \nu\overline{(\partial_k^2 v_i)v_j} + \overline{f_iv_j} + (i\leftrightarrow j) \\
&= -\partial_k\overline{v_iv_jv_k} - \partial_i\overline{pv_j} - \partial_j\overline{pv_i} + \overline{p(\partial_i v_j + \partial_j v_i)} \\
&+ \nu\partial_k^2\overline{v_iv_j} - 2\nu\overline{\partial_k v_i\partial_k v_j} + \overline{f_iv_j + f_jv_i}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\partial_t(\bar{v}_i\bar{v}_j) &= (\partial_t\bar{v}_i)\bar{v}_j + \bar{v}_i(\partial_t\bar{v}_j) \\
&= [-(\bar{v}_k\partial_k)\bar{v}_i - \partial_k\tau_{ik} - \partial_i\bar{p} + \nu\triangle\bar{v}_i + \bar{f}_i]\bar{v}_j + (i\leftrightarrow j) \\
&= -\partial_k(\bar{v}_i\bar{v}_j\bar{v}_k) - \partial_k[\tau_{ik}\bar{v}_j + \tau_{jk}\bar{v}_i] + \tau_{ik}(\partial_k\bar{v}_j) + \tau_{jk}(\partial_k\bar{v}_i) \\
&- \partial_i(\bar{p}\bar{v}_j) - \partial_j(\bar{p}\bar{v}_i) + \bar{p}(\partial_i\bar{v}_j + \partial_j\bar{v}_i) \\
&+ \nu\partial_k^2(\bar{v}_i\bar{v}_j) - 2\nu\partial_k\bar{v}_i\partial_k\bar{v}_j + (\bar{f}_i\bar{v}_j + \bar{f}_j\bar{v}_i)
\end{aligned}$$

Subtracting the two equations yields an equation for $\tau_{ij} = \overline{v_iv_j} - \bar{v}_i\bar{v}_j$.

To express the various terms that appear, we must introduce the generalized central moments of Germano (usually called cumulants in probability theory, or connected correlation functions in statistical physics). The nth-order generalized central moment $\tau(f_1,\ldots,f_n)$ is defined as follows:

$$\begin{aligned}
\bar{f}_1 &= \tau(f_1) \\
\overline{f_1f_2} &= \tau(f_1,f_2) + \bar{f}_1\bar{f}_2 \\
\overline{f_1f_2f_3} &= \tau(f_1,f_2,f_3) + \bar{f}_1\tau(f_2,f_3) + \bar{f}_2\tau(f_1,f_3) + \bar{f}_3\tau(f_1,f_2) + \bar{f}_1\bar{f}_2\bar{f}_3
\end{aligned}$$

and, iteratively,

$$\overline{f_1\ldots f_n} = \sum_{I\in\mathcal{P}}\prod_{j=1}^{p}\tau(f_{i_1^{(j)}},\ldots,f_{i_{n_j}^{(j)}})$$

where the sum is over the set $\mathcal{P}$ of all partitions $I = \{i_1^{(1)}, \ldots, i_{n_1}^{(1)}\}, \ldots, \{i_1^{(p)}, \ldots, i_{n_p}^{(p)}\}$ of the set $\{1, 2, ..., n\}$ with $\sum_{j=1}^{p} n_j = n$. We thus see that

$$\overline{f_1 \ldots f_n} = \tau(f_1, \ldots, f_n) + \text{terms defined by lower-order cumulant functions}$$

so that we may solve successively to obtain

$$\begin{aligned}
\tau(f_1) &= \bar{f}_1 \\
\tau(f_1, f_2) &= \overline{f_1 f_2} - \bar{f}_1 \bar{f}_2 \\
\tau(f_1, f_2, f_3) &= \overline{f_1 f_2 f_3} - \bar{f}_1 \tau(f_2, f_3) - \bar{f}_2 \tau(f_1, f_2) - \bar{f}_3 \tau(f_1, f_2) - \bar{f}_1 \bar{f}_2 \bar{f}_3 \\
&= \overline{f_1 f_2 f_3} - \bar{f}_1 \overline{f_2 f_3} - \bar{f}_2 \overline{f_1 f_3} - \bar{f}_3 \overline{f_1 f_2} + 2 \bar{f}_1 \bar{f}_2 \bar{f}_3
\end{aligned}$$

and etc.! Note: The "generalized central moments" $\tau(f_1, \ldots, f_n)$ are the cumulants of the random variables $f_1(\mathbf{x} + \mathbf{r}), \ldots, f_n(\mathbf{x} + \mathbf{r})$, distributed according to the density $G_\ell(\mathbf{r})$ on $\mathbf{r}$.

The final equation obtained for τ_{ij} has the form

$$\begin{aligned}
\partial_t \tau_{ij} + \partial_k J_{ijk} &= -[\bar{v}_{i,k} \tau_{kj} + \tau_{ik} \bar{v}_{j,k}] \leftarrow \text{production of stress by large-scale straining} \\
&+ \; 2\tau(p, S_{ij}) \leftarrow \text{pressure-strain correlation} \\
&- \; 2\nu \tau(v_{i,k}, v_{j,k}) \leftarrow \text{viscous destruction of stress} \\
&+ \; [\tau(v_i, f_j) + \tau(v_j, f_i)] \leftarrow \text{production of stress by forcing}
\end{aligned}$$

with $v_{i,k} = \frac{\partial v_i}{\partial x_k}$, etc. and

$$J_{ijk} = \tau_{ij} \bar{v}_k + \tau(p, v_i) \delta_{jk} + \tau(p, v_j) \delta_{ik} + \tau(v_i, v_j, v_k) - \nu \tau_{ij,k}$$

where

$$\begin{aligned}
\tau_{ij} \bar{v}_k &= \text{advective transport of stress} \\
\nu \tau_{ij,k} &= \text{viscous transport of stress}
\end{aligned}$$

Taking $\frac{1}{2}$ of the trace of the equation for τ_{ij} gives the equation for k, with

$$J_i^{E'} = k \bar{v}_i + \tau(p, v_i) + \frac{1}{2} \tau(v_k, v_k, v_i)$$

$$Q' \;=\; \tau(v_i, f_i)$$

Remark: There is a tempting analogy

	TURBULENCE	:	MOLECULAR DYNAMICS	
$\bar{e}_\ell = \frac{1}{2}\rho|\bar{\mathbf{v}}_\ell|^2$	large-scale kinetic energy	$\leftrightarrow$	$\frac{1}{2}\rho|\mathbf{v}|^2$	kinetic energy
ρk_ℓ	small-scale kinetic energy	$\leftrightarrow$	$u = \rho c_P T$	internal energy

For this reason, $\Pi_\ell = -\bar{S}_\ell : \boldsymbol{\tau}_\ell$ is sometimes called subscale dissipation (or "subgrid-scale dissipation" in LES). Note, however, that $\frac{1}{2}\rho|\mathbf{v}|^2 + u$ is conserved, while $\frac{1}{2}\rho|\bar{\mathbf{v}}_\ell|^2 + \rho k_\ell$ is NOT and, in fact, has the same space integral as total kinetic energy $\frac{1}{2}\rho|\mathbf{v}|^2$! (For this reason, a better correspondence is u for molecular dynamics and $\bar{u}^*_\ell = \bar{u}_\ell + \rho k_\ell$ for turbulence, so that the total $\frac{1}{2}\rho|\bar{\mathbf{v}}_\ell|^2 + \bar{u}^*_\ell$ is conserved.) Furthermore, there is a big separation in scale between the length L_∇ of variation of $\mathbf{v}$ and the mean-free-path λ_{mf} of molecules, whose energies (kinetic + potential) constitute u. As we discuss in more detail later, this is not true for $\bar{e}_\ell, k_\ell$.

There are some important alternative forms for the energy balances that we now discuss. Note that

$$\boldsymbol{\nabla}\bar{\mathbf{v}}_\ell : \boldsymbol{\tau}_\ell = \bar{\mathbf{v}}_\ell \cdot \mathbf{f}^s_\ell + \boldsymbol{\nabla}\cdot(\boldsymbol{\tau}_\ell \cdot \bar{\mathbf{v}}_\ell)$$

Where $\mathbf{f}^s_\ell = -\boldsymbol{\nabla}\cdot\boldsymbol{\tau}_\ell$. Thus, we may rewrite the energy balance as

$$\partial_t \bar{e}_\ell + \boldsymbol{\nabla}\cdot[(\bar{e}_\ell + \bar{p}_\ell)\bar{\mathbf{v}}_\ell - \nu\boldsymbol{\nabla}\bar{e}_\ell] = \bar{\mathbf{v}}_\ell \cdot \mathbf{f}^s_\ell - \nu|\boldsymbol{\nabla}\bar{\mathbf{v}}_\ell|^2 + \bar{\mathbf{v}}_\ell \cdot \bar{f}_\ell.$$

Where $\bar{\mathbf{v}}_\ell \cdot \mathbf{f}^s_\ell$ is the (negative) power input by the subscale force $\mathbf{f}^s_\ell$. Note, however, that this term is not Galilei invariant — an observer at rest and an observer moving with respect to a turbulent fluid would disagree about the "dissipation" due to such a term!

Another form of the balance can be written using the turbulent vortex-force

$$\mathbf{f}^v_\ell = \overline{(\mathbf{v}\times\boldsymbol{\omega})}_\ell - \bar{\mathbf{v}}_\ell \times \bar{\boldsymbol{\omega}}_\ell, \;\; f^v_{\ell i} = \epsilon_{ijk}\tau_\ell(v_j, \omega_k).$$

It is not hard to show using $\boldsymbol{\nabla}\cdot(\mathbf{v}\mathbf{v}) = \mathbf{v}\times\boldsymbol{\omega} - \boldsymbol{\nabla}(\frac{1}{2}|\mathbf{v}|^2)$ that

$$\mathbf{f}^s_\ell = \mathbf{f}^v_\ell - \boldsymbol{\nabla}k_\ell$$

so that

$$\partial_t \bar{\mathbf{v}}_\ell + (\bar{\mathbf{v}}_\ell \cdot \boldsymbol{\nabla})\bar{\mathbf{v}}_\ell = \mathbf{f}_\ell^v - \boldsymbol{\nabla}\bar{h}_\ell + \nu\triangle\bar{\mathbf{v}}_\ell + \bar{\mathbf{f}}_\ell$$

with $\bar{h}_\ell \equiv \bar{p}_\ell + k_\ell =$ "turbulent enthalphy". Then,

$$\partial_t \bar{e}_\ell + \boldsymbol{\nabla}\cdot[(\bar{e}_\ell + \bar{h}_\ell)\bar{\mathbf{v}}_\ell - \nu\boldsymbol{\nabla}\bar{e}_\ell] = \bar{\mathbf{v}}_\ell \cdot \mathbf{f}_\ell^v - \nu|\boldsymbol{\nabla}\bar{\mathbf{v}}_\ell|^2 + \bar{\mathbf{v}}_\ell \cdot \mathbf{f}_\ell$$

<u>Estimation of Energy Flux</u>

We have seen that the viscous dissipation in large-scale $\nu|\boldsymbol{\nabla}\bar{\mathbf{v}}_\ell|^2$ is negligible and that the energy flux

$$\Pi_\ell = -\boldsymbol{\nabla}\bar{\mathbf{v}}_\ell : \boldsymbol{\tau}_\ell$$

must therefore be the main "sink" term in the large-scale energy balance. We will now estimate this term. Note that

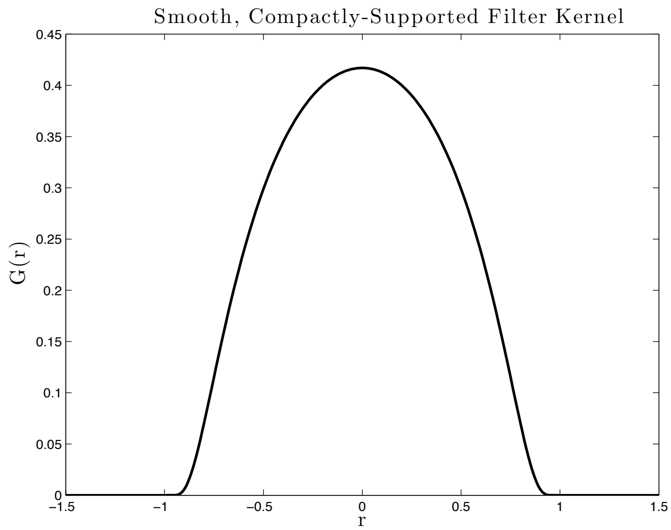
$$\begin{aligned} \boldsymbol{\nabla}\bar{\mathbf{v}}_\ell(\mathbf{x}) &= -\frac{1}{\ell}\int d^d r\ (\boldsymbol{\nabla}G)_\ell(\mathbf{r})\mathbf{v}(\mathbf{x}+\mathbf{r}) \\ &= -\frac{1}{\ell}\int d^d r\ (\boldsymbol{\nabla}G)_\ell(\mathbf{r})[\mathbf{v}(\mathbf{x}+\mathbf{r}) - \mathbf{v}(\mathbf{x})], \end{aligned}$$

since $\int d^d\mathbf{r}\, \boldsymbol{\nabla}G(\mathbf{r}) = 0$.

To get simple estimates, let us assume for the moment that G is C^∞ with compact support in the unit ball, e.g.

$$G(\mathbf{r}) = \begin{cases} N\exp[-\frac{1}{(1-r^2)}] & \text{for} \quad |r| \le 1 \\ 0 & \text{for} \quad \text{other } r \end{cases}$$

where $N \doteq 0.8822$ is a normalization factor for dimension $d = 3$.



We shall remove this restriction later on!

Then, with $|\mathbf{A}| = \sqrt{\sum_{ij} |a_{ij}|^2}$,

$$|\boldsymbol{\nabla}\bar{\mathbf{v}}_\ell(\mathbf{x})| \leq \frac{C}{\ell}\sup_{r<\ell}|\delta\mathbf{v}(\mathbf{r};\mathbf{x})| \ \text{ with } C = \int d^d\rho |\boldsymbol{\nabla} G(\rho)|$$

or

$$\boldsymbol{\nabla}\bar{\mathbf{v}}_\ell(\mathbf{x}) = O(\frac{\delta v(\ell;\mathbf{x})}{\ell})$$

with $\delta v(\ell;\mathbf{x}) \equiv \sup_{r<\ell}|\delta\mathbf{v}(\mathbf{r};\mathbf{x})|$. Now we must develop similar estimates for the stress $\boldsymbol{\tau}_\ell$. For this purpose, the following formula is crucial

$$\boxed{\tau_\ell(f,g) = \int d^d r\ G_\ell(\mathbf{r})\delta f(\mathbf{r})\delta g(\mathbf{r}) - \left[\int d^d r\ G_\ell(\mathbf{r})\delta f(\mathbf{r})\right]\left[\int d^d r\ G_\ell(\mathbf{r})\delta g(\mathbf{r})\right]}$$

or

$$\tau_\ell(f,g) = \langle \delta f \delta g\rangle_\ell - \langle \delta f\rangle_\ell \langle \delta g\rangle_\ell$$

where $\langle \cdot \rangle_\ell$ denotes average over $\mathbf{r}$ with respect to $G_\ell(\mathbf{r})$ and $\delta f(\mathbf{r};\mathbf{x}) = f(\mathbf{x}+\mathbf{r}) - f(\mathbf{x})$, etc. The above formula will turn out to be absolutely essential for much of our further analysis. It is one of the most useful formulas in the course, which we shall use many times. Amazingly, it was not discovered in this context until the 1990's [P. Constantin, et. al. *Commum. Math. Phys.*, **165**, 207 (1994); G. L. Eyink, *J. Stat.Phys.*, **78**, 335 (1995).]

It is easy to verify the formula by substituting the definitions of δf, δg and integrating. Later on, we shall prove a more general formula for all generalized central moments. Applying this formula to $\tau_{\ell ij} = \tau_\ell(v_i, v_j)$, we get

$$\begin{aligned} |\boldsymbol{\tau}_\ell(\mathbf{x})| &\leq \int d^d r\ G_\ell(\mathbf{r}) |\delta \mathbf{v}(\mathbf{r};\mathbf{x})|^2 + \left[\int d^d r\ G_\ell(\mathbf{r}) |\delta \mathbf{v}(\mathbf{r};\mathbf{x})| \right]^2 \\ &\leq \left[\int d^d r\ G_\ell(\mathbf{r}) \right] \delta v^2(\ell;\mathbf{x}) + \left[\int d^d r\ G_\ell(\mathbf{r}) \right]^2 \delta v^2(\ell;\mathbf{x}) \\ &= 2\delta v^2(\ell;\mathbf{x}), \end{aligned}$$

with $\int d^d r\ G_\ell(\mathbf{r}) = 1$, or,

$$\boldsymbol{\tau}_\ell(\mathbf{x}) = O(\delta v^2(\ell;\mathbf{x})).$$

Putting the two estimates together gives

$$|\Pi_\ell(\mathbf{x})| \leq (const.) \frac{\delta v^3(\ell;\mathbf{x})}{\ell}.$$

An estimate of this type was first derived by Lars Onsager around 1945 and closely related results were obtained by A. N. Kolomogorov in his famous papers in 1941, using probabilistic assumptions. The estimate has some important implications that we now discuss.

If $\mathbf{v}$ is continuously differentiable at point $\mathbf{x}$, then Taylor expansion in $\mathbf{r}$ gives

$$\begin{aligned} \mathbf{v}(\mathbf{x}+\mathbf{r}) &= \mathbf{v}(\mathbf{x}) + (\mathbf{r} \cdot \boldsymbol{\nabla}) \mathbf{v}(\mathbf{x}_\star) \\ \Longrightarrow \delta \mathbf{v}(\mathbf{r};\mathbf{x}) &= (\mathbf{r} \cdot \boldsymbol{\nabla}) \mathbf{v}(\mathbf{x}_\star) \\ \Longrightarrow \delta v(\ell;\mathbf{x}) &\leq \ell \cdot \sup_{\mathbf{x}} |\boldsymbol{\nabla} \mathbf{v}(\mathbf{x})| = O(\ell) \end{aligned}$$

In that case,

$$\Pi_\ell = O(\frac{\delta v^3(\ell)}{\ell}) = O(\ell^2) \to 0 \text{ as } \ell \to 0$$

Thus, Π_ℓ is too small for $\ell \leq \sqrt{\varepsilon / |\boldsymbol{\nabla} \mathbf{v}|^3}$ to account for a non-vanishing energy dissipation ε! More generally, suppose that

$$\delta v(\ell;\mathbf{x}) \sim \ell^h$$

for some $0 < h < 1$. Then,

$$\Pi_\ell = O(\frac{\delta v^3(\ell)}{\ell}) = O(\ell^{3h-1}) \text{ as } \ell \to 0 \text{ if } h > 1/3$$

This was first pointed out by Onsager(1945, 1949). A "minimal assumption" is that

$$\delta v(\ell) \sim (\varepsilon\ell)^{1/3},$$

which is the famous prediction of Kolmogorov & Obukhov(1941), Onsager(1945, 1949), and Heisenberg & von Weizsäcker(1948)

This scaling is often explained as the result of "dimensional analysis", but it has a deeper dynamic basis. We have already seen that $\mathbf{v}$ cannot remain differentiable in the limit that $\nu \to 0$, if the experiments are correct that

$$\varepsilon = \nu\langle|\boldsymbol{\nabla}\mathbf{v}|^2\rangle \nrightarrow 0 \text{ as } \nu \to 0.$$

There is a more refined result. We say that $\mathbf{v}$ is <u>Hölder continuous</u> at point $\mathbf{x}$ with exponent h, $0 < h < 1$, if

$$|\delta\mathbf{v}(\mathbf{r};\mathbf{x})| \leq C|\mathbf{r}|^h \qquad (\star)$$

for all $|\mathbf{r}| < r_0$ and some constant C. If it holds, then $\delta v(\ell;\mathbf{x}) = O(\ell)^h$. We thus see that

$(\star)$ cannot hold with $h > 1/3$ for all $\mathbf{x}$, as $\nu \to 0$, if the experiments are correct that $\varepsilon \nrightarrow 0$ in that limit (Onsager, 1949)

To make this argument a bit more convincing, we should consider the <u>total flux</u>

$$\int_V d^d\mathbf{x}\ \Pi_\ell(\mathbf{x}) \equiv \Pi_\ell,$$

or, equivalently, the <u>mean flux</u> over the flow domain

$$\langle\Pi_\ell\rangle = \frac{1}{|V|}\int_V d^3\mathbf{x}\ \Pi_\ell(\mathbf{x}).$$

Then

$$|\langle\Pi_\ell\rangle| \leq \langle|\Pi_\ell|\rangle = \frac{1}{|V|}\int_V d^3\mathbf{x}\ |\Pi_\ell(\mathbf{x})| = \|\Pi_\ell\|_1,$$

and

$$\|\Pi_\ell\|_1 \leq \|\Pi_\ell\|_r \text{ for } r \geq 1.$$

To get further estimates, we must recall some basic results for the L_p-norms, the

$$\begin{array}{rrcll} \underline{\text{Hölder inequality}}: & \|fg\|_1 & \leq & \|f\|_p\|g\|_q, & \frac{1}{p}+\frac{1}{q}=1 \\ \underline{\text{generalized Hölder inequality}}: & \|\prod_{i=1}^n f_i\|_r & \leq & \prod_{i=1}^n \|f_i\|_{p_i}, & \sum_{i=1}^n \frac{1}{p_i} = \frac{1}{r},\ r\geq 1 \end{array}$$

For example, see Kolmogorov & Fomin(1975), or any good textbook on real analysis. Since $\Pi_\ell = -\boldsymbol{\nabla}\bar{\mathbf{v}}_\ell : \boldsymbol{\tau}_\ell$,

$$\|\Pi_\ell\|_r \leq \|\boldsymbol{\nabla}\bar{\mathbf{v}}_\ell\|_{3r}\|\boldsymbol{\tau}_\ell\|_{3r/2}$$

To simplify notation, set $p = 3r$ or $r = p/3$, with $p \geq 3$. We must bound the terms $\|\boldsymbol{\nabla}\bar{\mathbf{v}}_\ell\|_p$, $\|\boldsymbol{\tau}_\ell\|_{p/2}$. Since

$$\boldsymbol{\nabla}\bar{\mathbf{v}}_\ell(\mathbf{x}) = -\frac{1}{\ell}\int d^d r\ (\boldsymbol{\nabla}G)_\ell(\mathbf{r})\delta\mathbf{v}(\mathbf{r};\mathbf{x}),$$

the triangle inequality gives

$$\|\boldsymbol{\nabla}\bar{\mathbf{v}}_\ell\|_p \leq \frac{1}{\ell}\int d^d r\ |(\boldsymbol{\nabla}G)_\ell(\mathbf{r})|\|\delta\mathbf{v}(\mathbf{r})\|_p.$$

Now, let us assume that for some σ_p, $0 < \sigma_p < 1$,

$$\|\delta\mathbf{v}(\mathbf{r})\|_p \leq C|\mathbf{r}|^{\sigma_p} \tag{$\star$}$$

for $|\mathbf{r}| \leq r_0$ and some constant $C > 0$. Then,

$$\begin{aligned} \|\boldsymbol{\nabla}\bar{\mathbf{v}}_\ell\|_p &\leq \frac{C}{\ell}\int d^d r\ |(\boldsymbol{\nabla}G)_\ell(\mathbf{r})|\cdot|\mathbf{r}|^{\sigma_p} \\ &= C'\ell^{\sigma_p-1} \quad (\text{substitute } \rho = \frac{\mathbf{r}}{\ell}) \end{aligned}$$

with $C' = C\int d^d\rho\ |\boldsymbol{\nabla}G(\rho)|\cdot|\rho|^{\sigma_p}$, which is assumed to be finite. Notice that we have <u>not</u> had to assume that G is compactly supported, but only that it decays rapidly enough for large $|\rho|$ so that the integral converges. This same approach could have been used earlier for the pointwise estimates. Thus,

$$\|\boldsymbol{\nabla}\bar{\mathbf{v}}_\ell\|_p = O(\ell^{\sigma_p - 1})$$

To estimate $\|\boldsymbol{\tau}_\ell\|_{p/2}$ we use the Hölder inequality again with

$$\boldsymbol{\tau}_\ell = \int d^3\mathbf{r} G_\ell(\mathbf{r})\delta\mathbf{v}(\mathbf{r})\delta\mathbf{v}(\mathbf{r}) - \int d^d r G_\ell(\mathbf{r})\delta\mathbf{v}(\mathbf{r}) \int d^d r G_\ell(\mathbf{r})\delta\mathbf{v}(\mathbf{r})$$

to get, for $p \geq 2$,

$$\begin{aligned} \|\boldsymbol{\tau}_\ell\|_{p/2} &\leq \int d^d r \; G_\ell(\mathbf{r})\|\delta\mathbf{v}(\mathbf{r})\delta\mathbf{v}(\mathbf{r})\|_{p/2} + \|\int d^d r \; G_\ell(\mathbf{r})\delta\mathbf{v}(\mathbf{r})\|_p^2 \\ &\leq \int d^d r G_\ell(\mathbf{r})\|\delta\mathbf{v}(\mathbf{r})\|_p^2 + \left[\int d^d r G_\ell(\mathbf{r})\|\delta\mathbf{v}(\mathbf{r})\|_p\right]^2 \\ &\leq C^2\left\{\int d^d r G_\ell(\mathbf{r})|\mathbf{r}|^{2\sigma_p} + \left[\int d^d r G_\ell(\mathbf{r})|\mathbf{r}|^{\sigma_p}\right]^2\right\} \\ &= C_p \ell^{2\sigma_p} \end{aligned}$$

where $C_p \equiv C^2[\int d^d\rho G(\rho)|\rho|^{2\sigma_p} + (\int d^d\rho G(\rho)|\rho|^{2\sigma_p})]$, which is assumed to be finite (a very modest requirement on G). Thus,

$$\|\boldsymbol{\tau}_\ell\|_{p/2} = O(\ell^{2\sigma_p}), \quad p \geq 2$$

and, finally,

$\|\Pi_\ell\|_p = O(\ell^{3\sigma_p - 1})$, $p \geq 3$, so that $\langle \Pi_\ell \rangle \to 0$ as $\ell \to 0$ unless $\sigma_p \leq \frac{1}{3}$ for $p \geq 3$.

We note that a function $\mathbf{v}$ with

$$\|\mathbf{v}\|_p < +\infty$$

and

$$\|\delta\mathbf{v}(\mathbf{r})\|_p \leq C|\mathbf{r}|^{\sigma_p}$$

is called Besov regular with pth-order Besov exponent σ_p. This is an "L_p-version" of Hölder continuity. Note that

$$\lim_{p\to\infty} \|\delta\mathbf{v}(\mathbf{r})\|_p = \|\delta\mathbf{v}(\mathbf{r})\|_\infty$$

$$= \sup_{x\in V} |\delta\mathbf{v}(\mathbf{r};\mathbf{x})|$$

Thus, the $p \to \infty$ limit of Besov regularity corresponds to uniform Hölder continuity, i.e.

$$|\delta\mathbf{v}(\mathbf{r};\mathbf{x})| \leq C|\mathbf{r}|^{\sigma_\infty} \text{ for all } \mathbf{x} \in V$$

for $|\mathbf{r}| \leq r_0$. Our previous result thus says that non-vanishing energy dissipation requires a velocity field which is not too regular in the limit $\nu \to 0$, i.e. $\mathbf{v}$ may not have $\sigma_p > \frac{1}{3}$ for any $p \geq 3$.

It is more traditional to consider so-called (absolute) structure functions for order p:

$$\begin{aligned} S_p(\mathbf{r}) &\equiv \langle|\delta\mathbf{v}(\mathbf{r})|^p\rangle \\ &= \|\delta\mathbf{v}(\mathbf{r})\|_p^p \end{aligned}$$

with assumed scaling exponent ζ_p

$$S_p(\mathbf{r}) \sim A_p u_{rms}^p (\frac{|\mathbf{r}|}{L})^{\zeta_p} \qquad (\star\star)$$

We have written this in a dimensionally correct form with $u_{rms} = \langle|\mathbf{v}-\langle\mathbf{v}\rangle|^2\rangle^{1/2}$ the root-square velocity and L a length-scale characteristic of the large-scale production mechanism. Note that

$$f(z) \sim g(z) \text{ for } z \ll 1$$

means that $\lim_{z\to 0} f(z)/g(z) = 1$. Then $(\star\star)$ implies the previous estimate $(\star)$ with $\sigma_p = \zeta_p/p$. Our previous result then implies that

$$\boxed{\langle\Pi_\ell\rangle \longrightarrow 0 \text{ for } \ell \ll L, \text{ unless } \zeta_p \leq \tfrac{p}{3} \text{ for } p \geq 3}$$

The classical Kolmogorov(1941) theory assumes that

$$\zeta_p = \tfrac{p}{3} \qquad \text{for all } p.$$

Then, using

$$\varepsilon \sim \frac{u_{rms}^3}{L},$$

one gets

$$S_p(\mathbf{r}) \sim C_p(\varepsilon|\mathbf{r}|)^{p/3} \text{ for all } p, \text{ and } |\mathbf{r}| \ll L.$$

Of course, only the inequality $\zeta_p \leq p/3$, $p \geq 3$ is rigorously implied. We shall discuss later the physical meaning of assuming that $\zeta_p = p/3$, but essentially, it is a "uniformity" assumption on velocity increments $|\delta\mathbf{v}(\mathbf{r};\mathbf{x})|$ which rules out large fluctuations in values for different $\mathbf{x}$.

Note that the estimate $\delta v(\ell) \sim (\varepsilon\ell)^{1/3}$ is consistent with

$$\langle \Pi_\ell \rangle \sim \frac{\delta v^3(\ell)}{\ell} \sim \varepsilon$$

for all $\ell \ll L$. Thus, one can explain the observed rate of energy dissipation ε — independent of viscosity ν — by the efficient transfer of energy down to small-scales where viscosity is effective. This length-scale is the so-called Kolmogorov (micro) scale η. It can be obtained as the length-scale at which

$$\Pi_\ell \sim \nu|\boldsymbol{\nabla}\bar{\mathbf{v}}_\ell|^2$$

We previously estimated the viscous dissipation by the *rms* velocity, but now we have an improved estimate in terms of velocity increments, as

$$\nu|\boldsymbol{\nabla}\bar{\mathbf{v}}_\ell|^2 = O(\frac{\nu\delta v^2(\ell)}{\ell^2})$$

Using $\Pi_\ell = O(\delta v^3(\ell))/\ell$, we get an estimate for η as the solution of

$$\frac{\delta v^3(\eta)}{\eta} \cong \nu\frac{\delta v^2(\eta)}{\eta^2} \Longrightarrow \delta v(\eta)\eta \cong \nu$$

Note that this implies that the "turbulent Reynolds number" $\frac{\delta v(\ell)\ell}{\nu}$ is approximately $\cong 1$ for $\ell \cong \eta$. If we now use the K41 (i.e. Kolmogorov 1941) conjecture that $\delta v(\eta) \sim (\varepsilon\eta)^{1/3}$, then

$$\varepsilon^{1/3}\eta^{4/3} \cong \nu \Longrightarrow \boxed{\eta \cong \nu^{3/4}\varepsilon^{-1/4}}$$

This is dimensionally correct, since $[\varepsilon] = L^2/T^3, [\nu] = L^2/T$. The K41 scaling $\delta v(\ell) \sim (\varepsilon\ell)^{1/3}$ is thus expected for a range of length-scales $\eta \ll \ell \ll L$, the so-called inertial (sub)range. This scaling prediction was one of the great early successes of turbulence theory, usually described in terms of energy spectra $E(k)$ in Fourier space. We do not discuss Fourier spectra here, but note only the rough equivalence $kE(k) \sim (\delta v(\ell))^2$ with $\ell \sim 1/k$.

The K41 prediction $E(k) \sim \varepsilon^{2/3}k^{-5/3}$ was finally verified in a rather convincing way by

H.L. Grant, R.W. Stewart & A. Moilliet, "Turbulent spectra from a tidal channel," J. Fluid Mech., **12**, 241-268 (1962).

Their data was presented at a famous meeting in Marseille in 1961, confirming the predictions of Kolmogorov twenty years earlier (1941). However, Kolmogorov himself was in attendance ... and he pointed out difficulties with his previous theory and proposed a "refinement" !!!

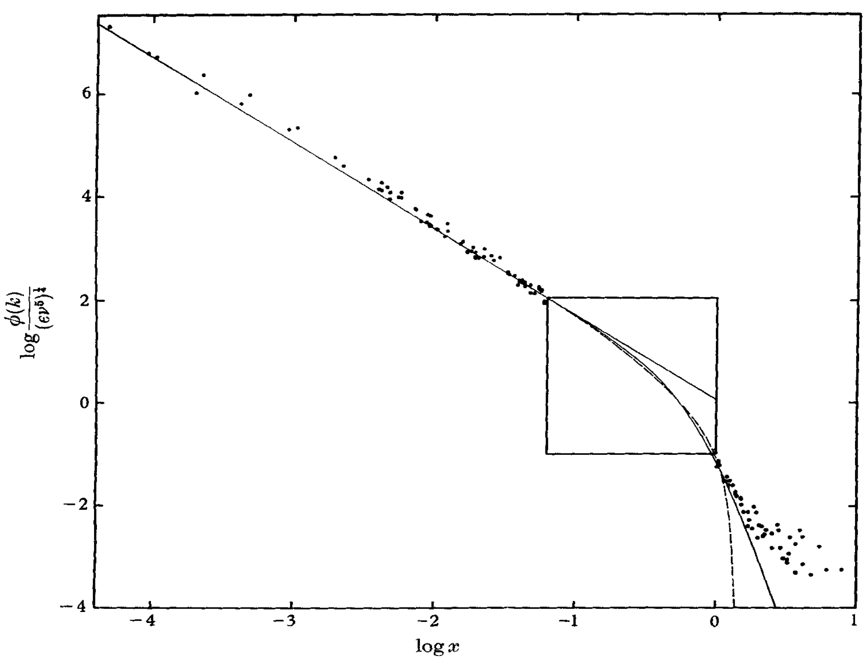
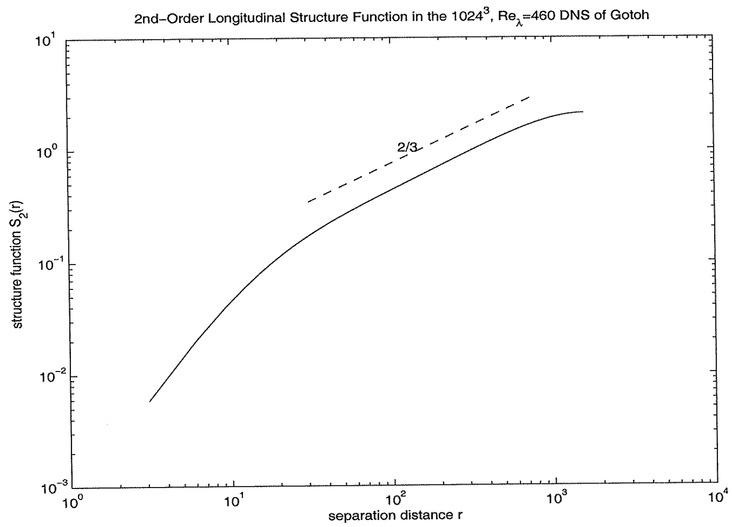
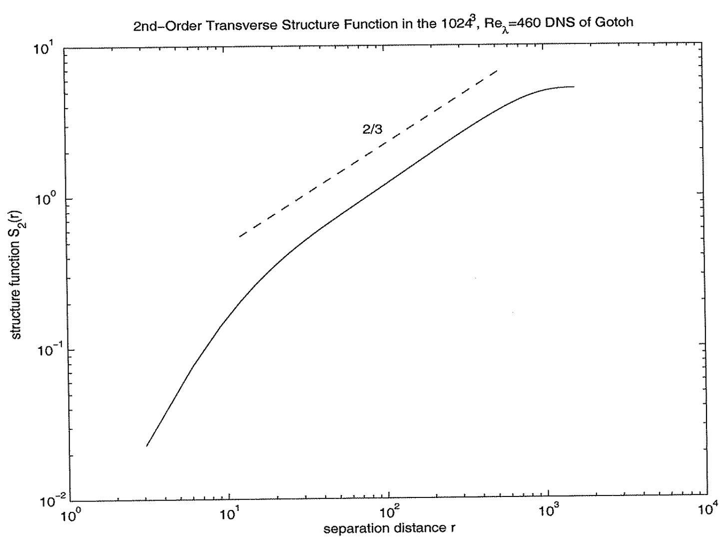


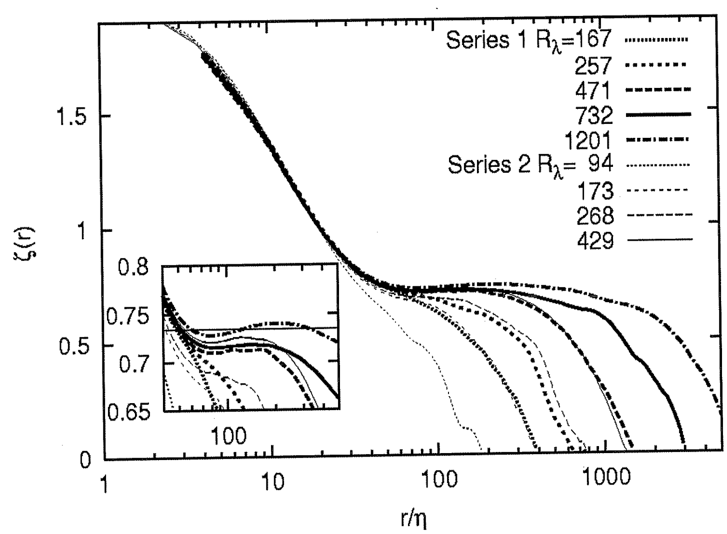
FIGURE 12. **Seventeen spectra compared to the theories of Kolmogoroff, Heisenberg and Kovasznay. The straight line has a slope of $-\frac{5}{3}$, the curved solid line is Heisenberg's theory and the dashed line is Kovasznay's theory. Within the square, the observations are too crowded to display on this scale and they are shown in figure 13.**

Nevertheless, the original K41 theory works well for $S_p(r)$ with p not too large ($p \approx 1 - 3$). See the plotted data from the 1024^3 DNS of T.Gotoh:





We also show results for the local slope $\zeta(r) = d\ln S_2(r)/d(\ln r)$, from the 4096^3 DNS on the Earth Simulator [Y. Kaneda et al., Phys. Fluids 15 (2): L21- L24 (2003)]



The inset is the enlargement of the range $40 < r/\eta < 500$. The straight line shows $\zeta(r) = 0.734$.

We now make a preliminary attempt to answer the question: Is the fluid approximation valid? If η is indeed the smallest length-scale in a turbulent flow, then

$$L_\nabla = \frac{|\mathbf{v}|}{|\boldsymbol{\nabla}\mathbf{v}|} \cong \eta.$$

The condition for validity of the hydrodynamic description is that (microscale) Knudsen number be small:

$$\frac{\lambda_{mf}}{\eta} \ll 1$$

Using

$$\begin{aligned} \lambda_{mf} &\cong \nu/c \text{ with } c \text{ the sound speed} \\ \eta &\cong \nu^{3/4}\varepsilon^{-1/4} \cong (\frac{\nu}{u_{rms}})^{3/4}L^{1/4} \text{ with } \varepsilon \cong \frac{u_{rms}^3}{L} \\ \Longrightarrow \frac{\lambda_{mf}}{\eta} &\cong (\frac{\nu}{u_{rms}L})^{1/4}(\frac{u_{rms}}{c}) = Ma(Re)^{-1/4} \end{aligned}$$

Thus, the fluid approximation is valid for

$$Ma \ll 1 \text{ and/or } Re \gg 1.$$

This argument goes back to S. Corrsin, "Outline of some topics in homogeneous turbulent flow," J. Geophys. Res.,**64**, 2134-2150(1959). It is important to note that validity of the Landau-Lifschitz fluctuating hydrodynamics requires the same condition.

Note that an important distinction is usually made between <u>turbulence</u> and <u>molecular dynamics</u>, due to a presumed <u>separation of scales</u> (both length and time) in the latter case, but NOT in the former. In derivations of hydrodynamics from deterministic Boltzmann equation, the signature of separation of length-scales is that a smooth "hydrodynamic profile" $\rho_a(\mathbf{x},t)$ exists, so that $\bar{\rho}_{a,\ell}(\mathbf{x},t) \approx \rho_a(\mathbf{x},t)$ for <u>all</u> ℓ in the range $\lambda_{mf} \ll \ell \ll L_\nabla$. The same is true for gradients $\boldsymbol{\nabla}\rho_a(\mathbf{x},t)$, time-derivatives $\partial_t\rho_a(\mathbf{x},t)$, etc. In the case of Boltzmann equation the coarse-grained stress tensor has the form $\overline{\mathbf{T}}_\ell = \rho\mathbf{v}\mathbf{v} + P\mathbf{I} - \eta[(\boldsymbol{\nabla}\mathbf{v}) + (\boldsymbol{\nabla}\mathbf{v})^\top]$ for <u>all</u> ℓ in the range $\lambda_{mf} \ll \ell \ll L_\nabla$. The dynamics is the same for all ℓ in this range.

In turbulence, however, all of the quantities such as $\bar{\mathbf{v}}_\ell$, $\mathbf{v}'_\ell$, $\boldsymbol{\nabla}\bar{\mathbf{v}}_\ell$, etc. change significantly throughout the whole range $(L_\nabla =)\eta \ll \ell \ll L$. For example, in K41 theory $\mathbf{v}'_\ell \sim (\langle\varepsilon\rangle\ell)^{1/3}$ and keeps growing as ℓ increases. We say that turbulent flows have a <u>continuous spectrum</u> of excitations in the range $\eta \ll \ell \ll L$. Among other things, this means that there is no law of large numbers in turbulence and $\bar{\mathbf{v}}_\ell$ is fluctuating, with dynamics that depends stochastically on unknown subscale modes $\mathbf{v}'_\ell$. The turbulent subscale stress for $\eta \ll \ell \ll L$ scales as

$$\boldsymbol{\tau}_\ell \sim (\delta v(\ell))^2 \sim (\langle\varepsilon\rangle\ell)^{2/3} \quad \text{(for K41)}$$

and is thus a monotonically increasing function of length-scale ℓ. Moreover, the turbulent stress is much more essentially stochastic. To see this, note that the turbulent stress at length-scale ℓ is not uniquely determined from the stress at scale $\ell' < \ell$ but instead (Germano, 1992):

$$\boldsymbol{\tau}_\ell(\mathbf{v},\mathbf{v}) \doteq \overline{(\boldsymbol{\tau}_{\ell'}(\mathbf{v},\mathbf{v}))}_\ell + \boldsymbol{\tau}_\ell(\overline{\mathbf{v}}_{\ell'},\overline{\mathbf{v}}_{\ell'})$$

To obtain the stress $\boldsymbol{\tau}_\ell$ requires knowledge not only of $\boldsymbol{\tau}_{\ell'}$ but also of the velocity field $\overline{\mathbf{v}}_{\ell'}$ resolved down to length-scale ℓ', which is unknown given only $\overline{\mathbf{v}}_\ell$.

In fact, however, we shall see that scale-separation is vitiated even in laminar flows by thermal fluctuations, so that gradients $\boldsymbol{\nabla}\bar{\rho}_{a,\ell}(\mathbf{x},t)$ become ℓ-dependent! In that case, the coarse-grained stress tensor has the form $\overline{\mathbf{T}}_\ell = \rho\mathbf{v}\mathbf{v} + P\mathbf{I} - \eta_\ell[(\boldsymbol{\nabla}\overline{\mathbf{v}}_\ell) + (\boldsymbol{\nabla}\overline{\mathbf{v}}_\ell)^\top] + \boldsymbol{\tau}'$ with $\boldsymbol{\tau}'$ the thermal noise arising as a central limit theorem correction to the law of large numbers and viscosity η_ℓ becomes ℓ-dependent. The distinction between turbulent and laminar is no longer so clear-cut.