Status and Interpretation of the Parisi-Frisch Multifractal Model

The Legendre transform

$$D(h) = \inf_p [ph + (d - \zeta_p)]$$

always exists and defines a concave function of h. To see the latter, note that for any $0 < \lambda < 1$ and h, h' fixed

$$[\lambda h + (1-\lambda)h']p + (d - \zeta_p) = \lambda[ph + (d - \zeta_p)] + (1-\lambda)[ph' + (d - \zeta_p)]$$

so that

$$\begin{aligned} D(\lambda h + (1-\lambda)h') &= \inf_p \{[\lambda h + (1-\lambda)h']p + (d - \zeta_p)\} \\ &\geq \lambda \inf_p [ph + (d - \zeta_p)] + (1-\lambda) \inf_p [ph' + (d - \zeta_p)] \\ &= \lambda D(h) + (1-\lambda) D(h'), \end{aligned}$$

which is the defining property of a concave function.

However, it is not clear which "fractal dimension" of $\mathcal{S}(h)$ that it represents (if any) or that it is even related to the singularity sets $\mathcal{S}(h)$ at all! These issues are discussed at some length in the cited papers of Eyink (1995) and Jaffard (1997). In particular, the latter paper constructs counterexamples, which show that there are multiscaling functions such that $D(h)$ defined as above from ζ_p does not give any of the common dimensions, such as $D_H(h), D_B(h)$, etc., for the singularity sets $\mathcal{S}(h)$. Jaffard also shows, however, that the Parisi-Frisch formalism is correct for certain classes of functions, such as self-similar functions. It is not known whether turbulent velocity fields (or other turbulent fields) possesses the required properties to justify the Parisi-Frisch formulism. It is an open mathematical issue.

However, a large number of predictions of the formula

$$\zeta_p = \inf_h [ph + (d - D(h))], \qquad (*)$$

are rigorous mathematical facts, even if the entire formula cannot be proved. We summarize some of these here:

(1) ζ_p is concave in p

This follows directly from $(*)$, but it has been proved rigorously earlier (using the Hölder inequality)

(2) $\zeta_p \sim ph_{min} + \kappa(h_{min}) + o(1)$

This follows from the general theory of Legendre transforms, applied to $(*)$. However, we have already discussed that

$$\sigma_p = \frac{\zeta_p}{p} \longrightarrow h_{min} \quad \text{as} \quad p \to \infty,$$

as an exact statement. Note that if $h_{min} = 0$, then (2) should be interpreted in the sense that

$$\zeta_p \sim o(p), \quad \text{as} \quad p \to \infty.$$

E.g. $\zeta_p \sim \sqrt{p}$, as $p \to \infty$ corresponds to $h_{min} = 0$.

(3) $D_H(h) \leq \inf_p[ph + (d - \zeta_p)]$

This follows from $(*)$ if either $D(h) = D_H(h)$ or if $D(h) = D_B(h)$. However, it can also be proved to be exactly true, as in Eyink (1995), Jaffard (1997). This results is useful because it shows how to put bounds on the sizes of singularity sets, using the scaling exponents ζ_p.

(4) $\sigma_p - \frac{d}{p} \geq \sigma_{p'} - \frac{d}{p'}$ for $p \geq p'$.

We first "prove" this inequality using $(*)$. Note that

$$\sigma_p = \frac{\zeta_p}{p} = \inf_h \{h + \frac{d-D(h)}{p}\}$$

so that

$$\sigma_p - \frac{d}{p} = \inf_h \{h - \frac{D(h)}{p}\}.$$

Since $D(h) \geq 0$, however, we see that this is an increasing (or, at least, non-decreasing) function of p. Thus, (4) "follows". In fact, (4) is a rigorous result, although it is usually stated differently as an embedding theorem for Besov spaces. In the standard language of function spaces, the embedding theorem states that

$$B_{p'}^{s',\infty}(\mathbb{R}^d) \subset B_p^{s,\infty}(\mathbb{R}^d)$$

for any $p \geq p'$ and $s - \frac{d}{p} < s' - \frac{d}{p'}$. For example, see H. Triebel, *Theory of Function Spaces* (Birkhäuser, 1983), Section 2.7. It is not hard to show that the above statement is equivalent to (4) [remembering that σ_p is the maximal Besov index of order p]. The inequality (4) can be extremely useful. E.g. it can be employed to show that Leray solutions of INS must satisfy $\langle |\mathbf{v}|^3 \rangle_{\text{space-time}} < +\infty$ and many other important results.

We have so far considered the geometric version of the Parisi-Frisch multifractal model, but there is another formulation, the probabilistic version. See Frisch (1995), Section 8.5.4 & 8.6.4 for a detailed discussion. In the probabilistic approach,

$$\zeta_p^P = \liminf_{r \to 0} \frac{\ln E(|\delta \mathbf{v}(\mathbf{r})|^p)}{\ln(r/L)} \qquad (*)$$

where E denotes average over a random ensemble. If this ensemble is space/time homogeneous, then $\zeta_p^P \leq \zeta_p^G$ almost surely (i.e. with probability one), where ζ_p^G is the geometric scaling exponent defined by a space/time average for each individual ensemble realization $\mathbf{v}$. If the above limit $(*)$ exists, then there is a theorem which states that the "scale-r Hölder exponent" defined by

$$h(\mathbf{r};\mathbf{x}) = \frac{\ln |\delta \mathbf{v}(\mathbf{r};\mathbf{x})|}{\ln(r/L)}$$

has the following "large deviations property"

$$\text{Prob}(h(\mathbf{r}) \approx h) \sim (\tfrac{r}{L})^{\kappa^P(h)}$$

where

$$\kappa^P(h) = \sup_p[\zeta_p^P - ph].$$

See R.S. Ellis, *Entropy, Large Deviations and Statistical Mechanics* (Springer, Berlin, 1985). This function $\kappa^P(h)$— the so-called "large-deviations rate function"—corresponds formally to the codimension $\kappa^G(h)$ in the geometric approach. However, it need no longer be true that $\kappa^P(h) \leq d$, i.e. it is possible to have negative "dimensions" $D^P(h) = d - \kappa^P(h) < 0$!

Multifractal Phenomenology

Of course, as physicists, we don't need to prove everything! Instead, we can use the multifractal model as a simple phenomenology from which to derive testable consequences for comparison with experiments & simulations (e.g. in the spirit of the "constituent quark model" in particle physics of strong nuclear forces). We shall consider several important predictions of this multifractal phenomenology in this section. In a later chapter, on Lagrangian dynamics & mixing, we shall examine more.

One important idea, proposed by

> G. Paladin & A. Vulpiani, "Anomalous scaling laws in multifractal objects," Phys. Reports **156** (4) 147-225 (1987)

is the notion of a fluctuating cut-off length η_h. This is the generalization of Kolmogorov dissipation length η to multifractal turbulence with a spectrum of singularities $[h_{min}, h_{max}]$, not just $h = \frac{1}{3}$. In this idea, at every point one balances energy flux and dissipation

$$\Pi_\ell(\mathbf{x}) \cong \nu|\boldsymbol{\nabla}\bar{\mathbf{v}}_\ell|^2 \quad \text{at} \quad \ell \cong \eta(\mathbf{x})$$

Using

$$\frac{\delta u^3(\ell)}{\ell} \cong \nu\frac{\delta u^2(\ell)}{\ell^2}$$

one obtains, as before,

$$\ell\delta u(\ell) \cong \nu,$$

or, with

$$\delta u(\ell) \sim v_0(\frac{\ell}{L})^h$$

$$v_0 L (\frac{\ell}{L})^{1+h} \cong \nu \Longrightarrow$$

$$\boxed{\ell \cong L(Re)^{-\frac{1}{1+h}} \equiv \eta_h}$$

with $Re = v_0 L/\nu$. Thus, the local dissipative cut-off $\eta(\mathbf{x})$ depends upon the value of the local Hölder exponent $h(\mathbf{x})$, and $\eta(\mathbf{x}) = \eta_h$ when $h(\mathbf{x}) = h$. For $h = \frac{1}{3}$ the above result reduces to $\eta_{1/3} = L \cdot (Re)^{-3/4} = \eta$, the K41 value.

This result has several interesting and (in principle) testable consequences. For example, it leads to the notion of an intermediate dissipation range, as elaborated by U. Frisch & M. Vergassola, "A prediction of the multifractal model: the intermediate dissipation range," Europhys. Lett. **14** 439-444(1991); see also, U. Frisch (1995), Section 8.5.5.

In this idea, the formula for the structure function $S_p(\mathbf{r}) = \langle|\delta\mathbf{v}(\mathbf{r})|^p\rangle$ is altered to

$$S_p(\mathbf{r}) \sim u_0^p \int_{\{h:\,\eta_h<r\}} d\mu(h)\ (\frac{r}{L})^{ph+\kappa(h)}$$

Then, for $L \gg r \gg \eta(h_*(p))$,

$$S_p(\mathbf{r}) \sim u_0^p (\tfrac{r}{L})^{\zeta_p}$$

just as before. However, for $\eta(h_*(p)) \gg r \gg \eta(h_{min})$, we expect that the contribution of h values from $h_*(p)$ to $h(r/L)$, given by $\eta(h(r/L)) \sim r$, will be cut out. Thus,

$$\begin{aligned}
S_p(\mathbf{r}) &\sim u_0^p (\frac{r}{L})^{[ph(\frac{r}{L})+\kappa(h(\frac{r}{L}))]} \\
\eta(h(\frac{r}{L})) &\equiv L \cdot (Re)^{\frac{-1}{1+h(r/L)}} \sim r \\
\Longrightarrow \ln S_p(\mathbf{r}) &\cong [ph(\frac{r}{L}) + \kappa(h(\frac{r}{L}))] \ln(\frac{r}{L}) \\
\frac{\ln(L/r)}{\ln Re} &= \frac{1}{1+h(r/L)} \equiv \theta
\end{aligned}$$

Taking the simultaneous limit $Re \to \infty, r \to 0$ with

$$\theta = \tfrac{\ln(L/r)}{\ln Re} \quad \text{fixed,}$$

then

$$\lim_{Re\to\infty, r\to 0, \theta \text{ fixed}} \frac{\ln S_p(\mathbf{r})}{\ln Re} = -[p\cdot h(p,\theta) + \kappa(h(p,\theta)))]\cdot\theta$$

with

$$h(p,\theta) = \begin{cases} h_*(p) & 0 \leq \theta < (1+h_*(p))^{-1} \\ \theta^{-1} - 1 & (1+h_*(p))^{-1} \leq \theta < (1+h_{min})^{-1} \end{cases}$$

This is an interesting parameter-free prediction that probes a range of range-scales on the border of the inertial range and the dissipation range (including scales smaller than the K41 cut-off η, if $h_{min} < \frac{1}{3}$.) Unfortunately, testing this prediction has proved quite challenging, since it is difficult for both experiments and simulations to resolve such small length-scales. For a recent attempt, see

R. Benzi et al., "Inertial range Eulerian and Lagrangian statistics from numerical simulations of isotropic turbulence," J. Fluid Mech **653** 221–244 (2010)

Another interesting set of predictions, of a similar character, are for velocity-gradient moments $\langle|\boldsymbol{\nabla}\mathbf{v}|^p\rangle$, or so-called velocity-gradient flatnesses

$$F_p = \frac{\langle|\boldsymbol{\nabla}\mathbf{v}|^p\rangle}{\langle|\boldsymbol{\nabla}\mathbf{v}|^2\rangle^{p/2}}$$

The key idea here is that of

M. Nelkin, "Multifractal scaling of velocity derivatives in turbulence," Phys. Rev. A **42** 1226–1229 (1990)

who proposed to estimate

$$|\boldsymbol{\nabla}\mathbf{v}(\mathbf{x})| \sim \frac{\delta u(\eta_h)}{\eta_h} \sim \frac{v_0}{L}(Re)^{\frac{1-h}{1+h}}$$

at points where $h(\mathbf{x}) = h$. This should happen at a fraction of points of the order of

$$\text{Fraction } \sim (\frac{\eta_h}{L})^{\kappa(h)} \sim (Re)^{\frac{-\kappa(h)}{1+h}}$$

Thus, for $Re \gg 1$,

$$\begin{aligned} \langle|\boldsymbol{\nabla}\mathbf{v}|^p\rangle &\sim (\frac{v_0}{L})^p \int_{h_{min}}^{h_{max}} d\mu(h)\, Re^{[\frac{p(1-h)-\kappa(h)}{1+h}]} \\ &\sim (\frac{v_0}{L})^p Re^{\gamma_p} \end{aligned}$$

with

$$\gamma_p = \sup_h [\tfrac{(1-h)p-\kappa(h)}{1+h}]$$

Frisch (1995), Section 8.6.5, describes a simple algorithm to determine the exponents γ_p. Here, we focus on the important issue whether the multifractal theory is consistent with

$$\nu\langle|\boldsymbol{\nabla}\mathbf{v}|^2\rangle \longrightarrow \varepsilon > 0 \quad \text{for} \quad Re \gg 1$$

which is equivalent to

$$\langle|\boldsymbol{\nabla}\mathbf{v}|^2\rangle \sim (\tfrac{v_0}{L})^2 \cdot Re \quad \text{for} \quad Re \gg 1$$

or $\gamma_2 = 1$. We now show that, in the multifractal model,

$$\boxed{\gamma_2 = 1 \iff \zeta_3 = 1}$$

This is completely consistent with the Kolmogorov 4/5-law! We establish this result assuming only that $h_{min} \geq -1$. The proof is then as follows:

$$\begin{aligned}
0 = \gamma_2 - 1 &= \sup_h [\frac{2-2h-\kappa(h)}{1+h} - 1] \\
&= \sup_h [\frac{1-3h-\kappa(h)}{1+h}] \\
\iff \frac{1-3h-\kappa(h)}{1+h} &\leq 0 \ \text{ for all } \ h \\
\text{and } \ \frac{1-3h_*-\kappa(h_*)}{1+h_*} &= 0 \ \text{ for some } \ h_* \\
\iff 1-3h-\kappa(h) &\leq 0 \ \text{ for all } \ h \\
\text{and } \ 1-3h_*-\kappa(h_*) &= 0 \ \text{ for some } \ h_* \\
\iff 0 &= \sup_h [1-3h-\kappa(h)] \\
&= 1 - \inf_h [3h+\kappa(h)] = 1-\zeta_3 \quad \text{QED!}
\end{aligned}$$

Notice that $h_* = \frac{d\zeta_p}{dp}|_{p=3}$ in this argument is the Hölder singularity that contributes all of the energy dissipation asymptotically in the limit $Re \to \infty$. The experimental/numerical results show that $h_* \lesssim \frac{1}{3}$.

This multifractal theory for velocity-gradients and fluctuating cutoffs is also consistent with some other exact PDE results, the theorem of Cafarelli, Kohn, Nirenberg (1982) on partial regularity of Leray solutions of INS. Those authors showed that the singularity set for Leray solutions is quite "small" in the precise sense that its (parabolic) Hansdorff dimension in space-time is at most one:

$$S = \text{ spacetime singularity set of INS solution } \mathbf{v}$$

$$D_H(S) \leq 1$$

Furthermore, the "length" of S (or 1-dimensional Hansdorff measure) is zero:

$$\mathcal{H}^1(S) = 0.$$

Finally CKN(1982) showed that $\mathbf{v} \to +\infty$ approaching the singularity set S in spacetime such that, essentially,

$$|\mathbf{v}(\mathbf{x}, t)| \geq \frac{(const.)}{\rho}$$

as $(\mathbf{x}, t) \to (\mathbf{x}_*, t_*) \in S$ and $\rho^2 = |\mathbf{x} - \mathbf{x}_*|^2 + \nu|t - t_*|$. In particular, as the singularity at $(\mathbf{x}_*, t_*)$ is approached at equal times ($t = t_*$), the velocity must blow up like r^{-1} which is a Hölder singularity with $h = -1$. (For more precise statements of this blow-up, see the original paper of CKN, 1982). We now show that these exact results are consistent with the multifractal formalism, as described above.

We note that the formula for the fluctuating cutoff should more properly be written as

$$\eta_h \sim L(\frac{Re}{Re_c})^{\frac{-1}{1+h}}$$

since the derivation neglected constants of proportionality. The constant Re_c has the interpretation of a "critical Reynolds number". Thus,

$$Re < Re_c \text{ and } h \geq -1$$

$$\implies \quad \eta_h \geq L$$

This is consistent with a known rigorous result that the Leray solution of INS is regular everywhere if the Reynolds number Re is sufficiently small! For example, see O. Ladyzhenskaya, *Mathematical Theory of Viscous Incompressible Flow* (Gordon & Breach, 1969). On the other hand,

$$Re > Re_c \quad \implies \quad \lim_{h \to -1} \eta_h = 0!$$

Thus, a real singularity ($\eta_h = 0$) may occur for Navier-Stokes, but only if there exists an $h \leq -1$. Note that the argument of CKN and that to derive the formula for η_h are both based on local energy balance.

This leads, finally, to the issue of the validity of the macroscopic hydrodynamic description in terms of INS, based on the multifracl model. Using the formula for η_h, it is easy to show that the result of Corrsin (1959) for K41 generalizes to

$$\frac{\eta_h}{\ell_{mf}} \sim (Re)^{\frac{h}{1+h}}/Ma$$

Thus, as long as $h_{min} > 0$, then $\eta_h \gg \ell_{mf}$ for $Re \gg 1$ and $Ma \ll 1$. It appears from this result that the macroscopic hydrodynamic description may break down if $h_{min} < 0$. However, the important result of

> J. Quastel & H.-T. Yau, "Lattice gases, large deviations, and the incompressible Navier-Stokes equations," Ann. Math. **148** 51-108 (1998)

is that (at least for a "toy model" of discrete space and discrete molecular velocities) there is some Leray solution of INS describing the coarse-grained velocity for $Kn \ll 1$ and $Ma \ll 1$, even if the Leray solution develops singularities! Since Leray solution are not unique if singular, molecular details may determine which is the physically correct Leray solution that applies.

Multifractal Model Based on Energy Dissipation

There is another version of the multifractal model based on energy dissipation

$$\varepsilon = \lim_{\nu \to 0} \nu |\boldsymbol{\nabla} \mathbf{v}^\nu|^2$$

This version arose historically even earlier than the Parisi-Frisch version, in the work of Mandelbrot, e.g.

> B. Mandelbrot, "Intermittent turbulence in self-similar cascades: divergence of high moments and dimension of the carrier," J. Fluid Mech. **62** 331-358 (1974)

We have already seen that, if $\mathbf{v}^\nu \to \mathbf{v}$ in spacetime L^3, then $\varepsilon = \lim_{\nu\to 0} |\boldsymbol{\nabla}\mathbf{v}^\nu|^2$ exists as a positive measure which assigns a non-negative number to each (measurable) set $\Delta \subset \mathbb{R}^d$

$$\varepsilon : \Delta \longmapsto \varepsilon(\Delta) \geq 0,$$

that is, finally, [2]

$$\varepsilon(\Delta) = \int_\Delta \varepsilon(\mathbf{x}) d^d x$$

Now suppose that $\Delta_r(\mathbf{x})$ is a square of sidelength r centered at $\mathbf{x}$, or a ball of radius r centered at $\mathbf{x}$. Then we say that the measure ε is Hölder continuous with exponent α at point $\mathbf{x}$, if

$$\varepsilon(\Delta_r(\mathbf{x})) = O((\tfrac{r}{L})^\alpha), \qquad r \ll L$$

Similarly, we may define the (maximal) Hölder exponent $\alpha(\mathbf{x})$ at point $\mathbf{x}$ by

$$\alpha(\mathbf{x}) = \liminf_{r \to 0} \frac{\ln \varepsilon(\Delta_r(\mathbf{x}))}{\ln(r/L)} = \liminf_{r \to 0} \alpha_r(\mathbf{x}).$$

Every part of the multifractal model for velocity increments may be carried over, in perfect analogy, to the dissipation measure. Thus, the fraction of space on which $\alpha_r(\mathbf{x})$ takes the value α is determined by the codimension $\kappa(\alpha)$ of the set, or, more conventionally, by its negative $f(\alpha) = -\kappa(\alpha)$:

$$\text{Fraction}(\{\mathbf{x} : \alpha_r(\mathbf{x}) \approx \alpha\}) \;\sim\; (\tfrac{r}{L})^{-f(\alpha)} \quad \text{for} \quad r \ll L$$

In that case, one can expect that

$$< [\varepsilon(\Delta_r)]^p > \sim \; \langle \varepsilon \rangle^p (\tfrac{r}{L})^{\tau_p} \quad \text{for} \quad r \ll L$$

with[3]

$$\begin{aligned} \tau_p &= \inf_\alpha \{\alpha p - f(\alpha)\}, \\ f(\alpha) &= \inf_p [p\alpha - \tau_p]. \end{aligned}$$

Considerable evidence has been obtained in experiments and simulations for multifractality of the energy dissipation. See

C. M. Meneveau & K. R. Sreenivasan, "The multifractal spectrum of the dissipation field in turbulent flows," Nucl. Phys. B, Proc. Suppl. **2** 49-76 (1987)

[2] Rigorously, ε has only been proved to be a spacetime measure. Thus $\varepsilon(\Delta)$ should really be interpreted as averaged over a small time-interval $[t - \tau, t + \tau]$ around t.

[3] It is worthwhile noting that Mandelbrot (1974) did not introduce either α or $f(\alpha)$. Instead he employed a different notion of "generalized dimensions" defined by $D_p = \tau_p/(p-1)$ for each p.

C. M. Meneveau & K. R. Sreenivasan, "The multifractal nature of turbulent energy dissipation," J. Fluid Mech. **224** 429-484 (1991)

There is also a conjectured relation between the multifractal approaches, the Kolmogorov refined similarity hypothesis. This subject properly belongs to the statistical approach, so we say only a few words about it here. The basic paper is

A. N. Kolmogorov, "A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds number," J. Fluid Mech. **13** 82-85(1962)

Kolmogorov proposed that the random variable

$$W_r(\mathbf{x}) \equiv \frac{|\delta\mathbf{v}(\mathbf{r};\mathbf{x})|}{[\varepsilon(\Delta_r(\mathbf{x}))r]^{1/3}}$$

conditioned on $\varepsilon(\Delta_r(\mathbf{x}))$ is independent of $\varepsilon(\Delta_r(\mathbf{x}))$ when the "local Reynolds number" $Re_r(\mathbf{x}) \equiv \varepsilon(\Delta_r(\mathbf{x}))^{1/3}r^{4/3}/\nu$ is $\gg 1$ and has a universal distribution independent of r. In that case, the RSH implies

$$\begin{aligned}\langle|\delta\mathbf{v}(\mathbf{r})|^p\,|\,\varepsilon(\Delta_r)\rangle &= \langle W_r^p\,|\,\varepsilon(\Delta_r)\rangle[\varepsilon(\Delta_r)]^{p/3} \\ &\approx \langle W^p\rangle[\varepsilon(\Delta_r)r]^{p/3}\end{aligned}$$

for $Re_r \gg 1$, so that

$$\begin{aligned}\langle|\delta\mathbf{v}(\mathbf{r})|^p\rangle &= \langle|\delta\mathbf{v}(\mathbf{r})|^p\,|\,\varepsilon(\Delta_r)\rangle \\ &\approx C_p\left\langle\Big[\varepsilon(\Delta_r)\Big]^{p/3}\right\rangle r^{p/3},\;\; C_p=\langle W^p\rangle \\ &\sim (\langle\varepsilon\rangle r)^{p/3}(\frac{r}{L})^{\tau_{p/3}} \\ &\sim u_0^p(\frac{r}{L})^{\zeta_p}\end{aligned}$$

with

$$\zeta_p = p/3 + \tau_{p/3}$$

This relation has met with some empirical success (e.g. see Meneveau & Sreenivasan, 1991). However, many theoretical criticisms have been levelled against the RSH. See Frisch(1995), Section 8.6.2 for a survey & discussion.

(F) Scale-Locality Revisited

We are now in a position to review the issue of scale-locality of turbulent energy cascade, in light of our understanding of intermittency & anomalous scaling. Our previous local-in-space estimates were subject to a criticism that velocity increments $\delta\mathbf{v}(\mathbf{r};\mathbf{x})$ need not show scaling behavior locally. In fact, experiments & simulations show that $\delta\mathbf{v}(\mathbf{r};\mathbf{x})$ at a point $\mathbf{x}$ tends to oscillate rather wildly for $r \ll L$ and does not show clean scaling. (This is one of the reasons that more sophisticated methods than structure functions are required to get good estimates for $D(h)$-spectra, e.g. the maximum-modulus wavelet method.) However, structure-functions themselves seem to have very good scaling

$$S_p(\mathbf{r}) = \langle|\delta\mathbf{v}(\mathbf{r})|^p\rangle \sim u_0^p(\tfrac{r}{L})^{\zeta_p}$$

we therefore can obtain more controlled estimates on fractional contibutions by using global L_p estimates rather than local-in-space estimates; i.e., good estimates hold for quantities such as

$$\frac{\|\delta\bar{\mathbf{v}}_\Delta(\mathbf{r})\|_p}{\|\delta\mathbf{v}(\mathbf{r})\|_p},\ \Delta \gg r \qquad \text{or} \qquad \frac{\|\delta\mathbf{v}'_\delta(\mathbf{r})\|_p}{\|\delta\mathbf{v}(\mathbf{r})\|_p},\ \delta \ll r$$

Let us state a precise result for velocity increments:

Lemma. If $\mathbf{v}$ satisfies for some exponent $0 < \sigma_p < 1$ the scaling law

$$\|\delta\mathbf{v}(\mathbf{r})\|_p \sim r^{\sigma_p} \quad \text{as} \quad r \to 0$$

then the velocity increment is IR-local in the L_p-sense: for $\Delta \gg r$,

$$\|\delta\mathbf{v}(\mathbf{r}) - \delta\mathbf{v}'_\Delta(\mathbf{r})\|_p = \|\delta\bar{\mathbf{v}}_\Delta(\mathbf{r})\|_p = O(r\Delta^{\sigma-1}) = \|\delta\mathbf{v}(\mathbf{r})\|_p \cdot O((\tfrac{r}{\Delta})^{1-\sigma_p})$$

and also UV-local in the L_p-sense: for $\delta \ll r$,

$$\|\delta\mathbf{v}(\mathbf{r}) - \delta\bar{\mathbf{v}}_\delta(\mathbf{r})\|_p = \|\delta\mathbf{v}'_\delta(\mathbf{r})\|_p = O(\delta^{\sigma_p}) = \|\delta\mathbf{v}(\mathbf{r})\|_p \cdot O((\tfrac{\delta}{r})^{\sigma_p})$$

The proof is essentially identical to that which we gave before, but replacing local bounds by global L_p bounds and local Hölder conntinuity by global scaling.

Since quantities such as coarse-grained velocity-gradients $\bar{\mathbf{D}}_\ell(\mathbf{v}) = \boldsymbol{\nabla}\bar{\mathbf{v}}_\ell$ and stress $\boldsymbol{\tau}_\ell(\mathbf{v},\mathbf{v})$ can all be expressed in terms of velocity increments, the basic locality properties extend directly to them as well. For example, for coarse-grained velocity gradients, with $p \geq 1$:

IR-locality:

$$\|\bar{\mathbf{D}}_\ell(\mathbf{v}) - \bar{\mathbf{D}}_\ell(\mathbf{v}'_\Delta)\|_p = \|\bar{\mathbf{D}}_\ell(\bar{\mathbf{v}}_\Delta)\|_p = O(\Delta^{\sigma_p - 1})$$

UV-locality:

$$\|\bar{\mathbf{D}}_\ell(\mathbf{v}) - \bar{\mathbf{D}}_\ell(\mathbf{v}'_\delta)\|_p = \|\bar{\mathbf{D}}_\ell(\mathbf{v}'_\delta)\|_p = O(\tfrac{\delta^{\sigma_p}}{\ell})$$

and for subscale stress, with $p \geq 2$:

IR-locality:

$$\|\boldsymbol{\tau}_\ell(\mathbf{v},\mathbf{v}) - \boldsymbol{\tau}_\ell(\mathbf{v}'_\Delta,\mathbf{v})\|_{p/2} = \|\boldsymbol{\tau}_\ell(\bar{\mathbf{v}}_\Delta,\mathbf{v})\|_{p/2} = O(\ell^{\sigma_p+1}\Delta^{\sigma_p-1})$$

UV-locality:

$$\|\boldsymbol{\tau}_\ell(\mathbf{v},\mathbf{v}) - \boldsymbol{\tau}_\ell(\bar{\mathbf{v}}_\delta,\mathbf{v})\|_{p/2} = \|\boldsymbol{\tau}_\ell(\mathbf{v}'_\delta,\mathbf{v})\|_{p/2} = O(\ell^{\sigma_p}\delta^{\sigma_p})$$

Notice that in order to obtain these estimates for the stress we had to use the Hölder inequality to bound $L^{p/2}$ norms by a product of two L^p norms, e.g. $\|\delta\mathbf{v}(\mathbf{r})\delta\mathbf{v}(\mathbf{r})\|_{p/2} \leq \|\delta\mathbf{v}(\mathbf{r})\|_p^2$.

These estimates already tell us that the contributions of modes distant in scale — either $\Delta \gg \ell$ or $\delta \ll \ell$ — are small and vanish in the limits that $\ell/\Delta \to 0$ or $\delta/\ell \to 0$. We can further estimate the relative contributions, if we know something about the scaling of the main part, e.g.

$$\langle|\boldsymbol{\tau}_\ell(\mathbf{v},\mathbf{v})|^{p/2}\rangle \;\sim\; u_0^p(\tfrac{\ell}{L})^{\rho_{p/2}}$$

Since $|\boldsymbol{\tau}_\ell(\mathbf{v},\mathbf{v})| \sim \delta u^2(\ell)$ one might expect, heuristically, that

$$\langle|\boldsymbol{\tau}_\ell(\mathbf{v},\mathbf{v})|^{p/2}\rangle \;\sim\; \langle\delta u^p(\ell)\rangle u_0^p(\tfrac{\ell}{L})^{\zeta_p}$$

and, thus,

$$\rho_{p/2} = \zeta_\rho.$$

In fact, this relation works quite well. For experimental evidence, see

C. Meneveau & J. O'Neil, "Scaling laws of the dissipation rate of turbulent subgrid-scale kinetic energy," Phys. Rev. E **49** 2866-2874(1994)

In that case, $\|\boldsymbol{\tau}_\ell(\mathbf{v},\mathbf{v})\|_{p/2} \sim \ell^{2\sigma_p}$, so that one obtains

IR-locality: $\|\boldsymbol{\tau}_\ell(\bar{\mathbf{v}}_\Delta,\mathbf{v})\|_{p/2} \sim \|\boldsymbol{\tau}_\ell(\mathbf{v},\mathbf{v})\|_{p/2} \cdot O((\tfrac{\ell}{\Delta})^{1-\sigma_p})$

UV-locality: $\|\boldsymbol{\tau}_\ell(\mathbf{v}'_\delta,\mathbf{v})\|_{p/2} \sim \|\boldsymbol{\tau}_\ell(\mathbf{v},\mathbf{v})\|_{p/2} \cdot O((\tfrac{\delta}{\ell})^{\sigma_p})$

Similar consideration apply to the energy flux

$$\Pi_\ell(\mathbf{v},\mathbf{v},\mathbf{v}) = -\bar{\mathbf{D}}_\ell(\mathbf{v}) : \boldsymbol{\tau}_\ell(\mathbf{v},\mathbf{v})$$

which depends on three velocity modes. Since, heuristically,

$$\Pi_\ell \sim \frac{\delta u^3(\ell)}{\ell}$$

we might expect that

$$\langle |\Pi_\ell|^{p/3} \rangle \sim \ell^{\tau_{p/3}}$$

with

$$\langle |\Pi_\ell|^{p/3} \rangle \sim \langle \delta u^p(\ell) \rangle / \ell^{p/3} \sim \ell^{\zeta_p - \frac{p}{3}}$$

so that

$$\tau_{p/3} = \zeta_p - p/3$$

or

$$\zeta_p = p/3 + \tau_{p/3}$$

This looks very similar to the result obtained from Kolmogorov RSH, except that here $\tau_{p/3}$ is the scaling of energy flux not volume-integrated energy dissipation. In fact, the above relation was proposed by Kraichnan as an alternative to Kolmogorov's relation:

R. H. Kraichnan, "On Kolmogorov's inertial-range theories," J. Fluid Mech. **62** 305-330 (1974)

This is thus called Kraichnan's refined similarity hypothesis (RSH). For empirical evidence in favor of this relation, see

S. Cerutti & C. Meneveau, "Intermittency and relative scaling of subgrid scale energy dissipation in isotropic turbulence," Phys. Fluids **10** 928-937 (1998)

Q. Chen et al. "Intermittency in the joint cascade of energy and helicity," Phys. Rev. Lett. **90** 214503 (2003)

This result may also be stated as

$$\|\Pi_\ell\|_{p/3} \sim \ell^{3\sigma_p - 1}, \;\; p \geq 3$$

In that case, we can make similar locality statements about the energy flux itself, such as

IR-locality: $\|\Pi_\ell(\bar{\mathbf{v}}_\Delta, \mathbf{v}, \mathbf{v})\|_{p/3} = \|\Pi_\ell(\mathbf{v}, \mathbf{v}, \mathbf{v})\|_{p/3} \cdot O((\frac{\ell}{\Delta})^{1-\sigma_p})$

UV-locality: $\|\Pi_\ell(\mathbf{v}'_\delta, \mathbf{v}, \mathbf{v})\|_{p/3} = \|\Pi_\ell(\mathbf{v}, \mathbf{v}, \mathbf{v})\|_{p/3} \cdot O((\frac{\delta}{\ell})^{\sigma_p})$

In these estimates one might replace any of the three $\mathbf{v}$'s by a $\bar{\mathbf{v}}_\Delta$ or $\mathbf{v}'_\delta$ and the bound will be the same. Replacing two or more $\mathbf{v}$'s will lead to even smaller bounds. Particularly interesting is the case that $p = 3$, because

$$|\int d^dx \; \Pi_\ell(\mathbf{x})| \leq \int d^dx |\Pi_\ell(\mathbf{x})| = \|\Pi_\ell\|_1$$

bounds the total space-integrated energy flux to small scales. Thus, employing the above bounds for $p = 3$ we see that the mean energy flux over the whole domain is dominated by interactions that are local in scale, as long as

$$0 < \sigma_3 < 1.$$

This must be true as long as there is a persistent energy cascade at all, since we know that then $\sigma_3 \cong 1/3$.

The estimates for $p > 3$ give stronger bounds on the scale-nonlocal contributions, showing that not only average flux is dominated by modes local-in-scale but also pth-order moments of flux are dominated by local modes. In the limit $p \to \infty$ these estimates become bounds on nonlocal contributions to flux uniformly in space. However, we know that σ_p is decreasing and

$$\sigma_p \searrow h_{min} \quad \text{as} \quad p \to \infty.$$

Thus, the IR-bounds become better for larger p, but the UV-bounds become worse. As $p \to \infty$, the moments $\langle |\Pi_\ell|^{p/3} \rangle$ are more and more dominated by the most singular events for which transfer is less local. However, it is reassuring that strong forms of scale-locality survive even in the presence of intense intermittency. For more on this topic, see

G. Eyink, Physica D **207** 91-116 (2005)