(D) Empirical Results on Turbulence Scaling of Velocity Increments

We shall present here a selection of the empirical studies of inertial-range scaling of velocity increments by laboratory experiment and numerical simulation. We begin with

⋆ F. Anselmet et al.,"High-order velocity structure functions in turbulent shear flow," J. Fluid Mech. **140** 63-89(1984)

This paper studied the longitudinal velocity structure functions

$$S_p^L(r) = \langle [\delta v_L(\mathbf{r})]^p \rangle \sim C_p^L \ell^{\zeta_p^L}$$

and found good evidence of derivations from K41 for higher-order values of p. It was historically important in stimulating a great deal of work to understand the exponents ζ_p^L.

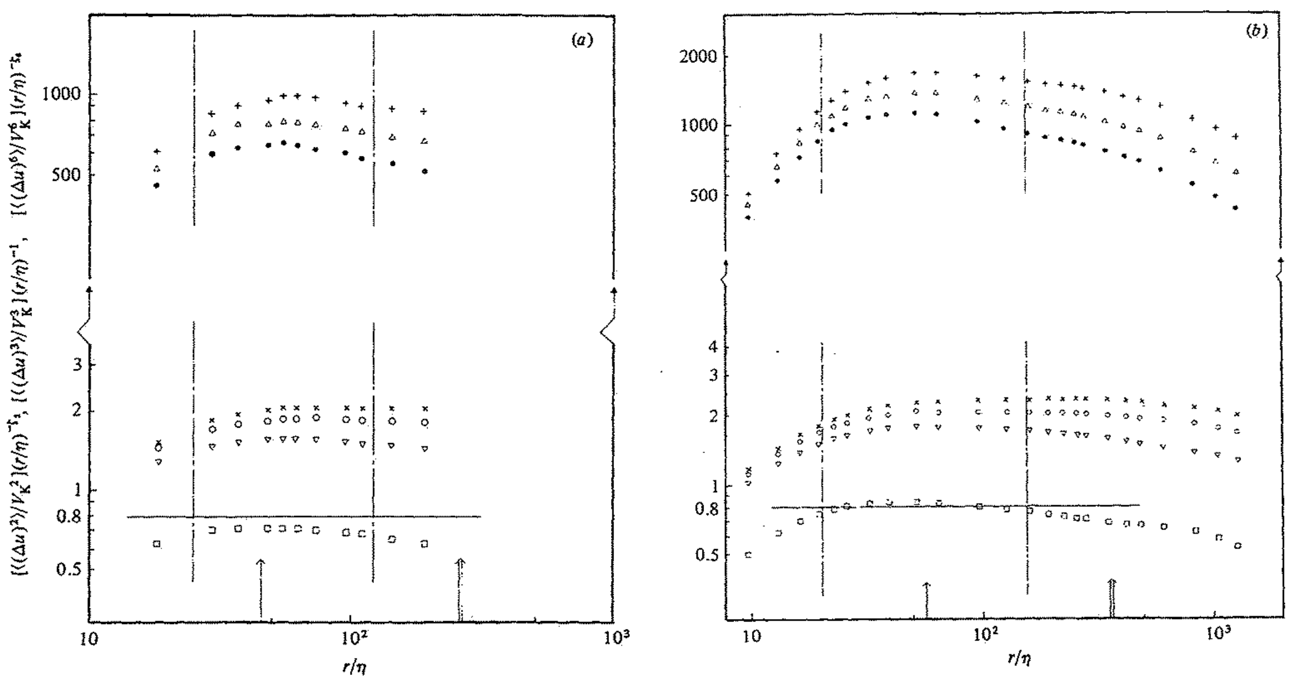


FIGURE 10. Structure functions of second-, third- and sixth-order multiplied by appropriate powers of r/η, with r/η in the jet: □, $[\langle(\Delta u)^3\rangle/V_K^3]\,(r/\eta)^{-1}$; ×, $[\langle(\Delta u)^2\rangle/V_K^2]\,(r/\eta)^{-\frac{2}{3}}$; ○, $(r/\eta)^{-0.69}$ (LN with $\mu = 0.2$); ▽, $(r/\eta)^{-0.73}$ (β-model, $\mu = 0.2$); +, $[\langle(\Delta u)^6\rangle/V_K^6]\,(r/\eta)^{-1.75}$; △, $(r/\eta)^{-1.8}$; ●, $(r/\eta)^{-1.85}$. (a) $R_\lambda = 536$. (b) $R_\lambda = 852$. The vertical lines –·–·– indicate the inertial-range limits, the vertical arrow the Taylor microscale, and the double vertical arrow indicates $\frac{1}{4}L$.

n	2	3	4	5	6	7	8	9	10	12	14	16	18
$R_\lambda = 515$ (duct)	0.71	1	1.33	—	1.8	—	2.27	—	2.64	2.94	3.32	—	—
$R_\lambda = 536$ (jet)	0.71	1	1.33	1.54	1.8	2.06	2.28	2.41	2.60	2.74	—	—	—
$R_\lambda = 852$ (jet)	0.71	1	1.33	1.65	1.8	2.12	2.22	2.52	2.59	2.84	3.28	3.49	3.71

TABLE 2. Values of the exponent ζ_n for $2 \leqslant n \leqslant 18$

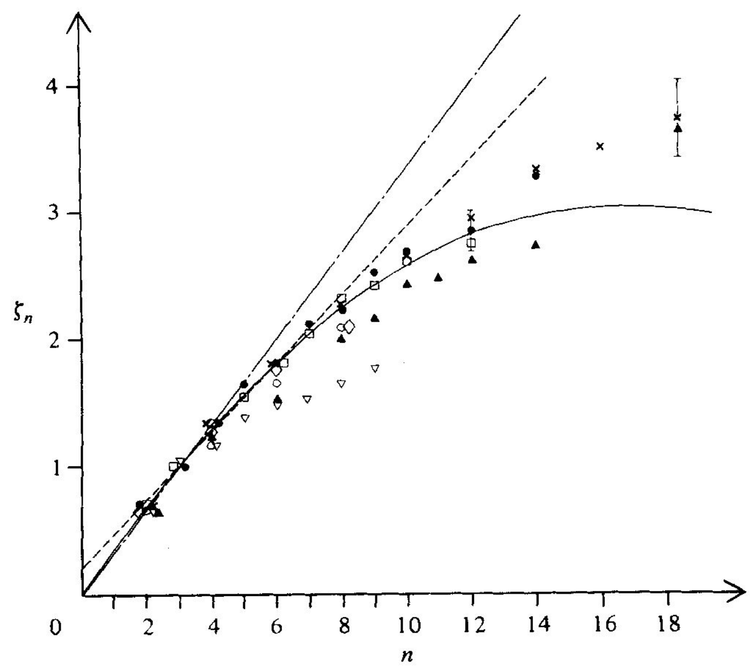


FIGURE 14. Variation of exponent ζ_n as a function of the order n. ●, $R_\lambda = 515$ (duct); □, 536; ×, 852. Symbols ○, ▲, ▽, ◇ are respectively the exponents given by Mestayer (1980); Vasilenko *et al.* (1975); Van Atta & Park (1972); and Antonia *et al.* (1982*a*). The solid curve is LN with $\mu = 0.2$, the dotted curve the β-model and the chain-dotted line Kolmogorov's (1941) model.

$\star$ K. R. Sreenivasan et al., "Asymmetry of velocity increments in fully developed turbulence and the scaling of low-order moments," Phys. Rev. Lett. **77** 1588-1491 (1996)

These authors considered also absolute longitudinal structure functions

$$S_p^{AL}(r) = \langle|\delta v_L(\mathbf{r})|^p\rangle \sim C_p^{AL}\ell^{\zeta_p^{AL}}$$

and structure functions for positive and negative parts

$$\delta u_L^{\pm}(\mathbf{r}) = \tfrac{|\delta v_L(\mathbf{r})|\pm\delta v_L(\mathbf{r})}{2} \geq 0$$

so that $\delta v_L(\mathbf{r}) = \delta u_L^+(\mathbf{r}) - \delta u_L^-(\mathbf{r})$ and

$$S_p^{\pm}(\mathbf{r}) = \langle[\delta u_L^{\pm}(\mathbf{r})]^p\rangle \sim C_p^{\pm L}\ell^{\zeta_p^{\pm L}}.$$

Since $|\langle[\delta v_L(\mathbf{r})]^p\rangle| \leq \langle|\delta v_L(\mathbf{r})|^p\rangle$, it is easy to see that

$$\zeta_p^{AL} \leq \zeta_p^L.$$

The paper reports evidence that

$$\zeta_p^L \cong \zeta_p^{AL} \cong \zeta_p^{-L} < \zeta_p^{+L} \quad \text{for} \quad p > 1.$$

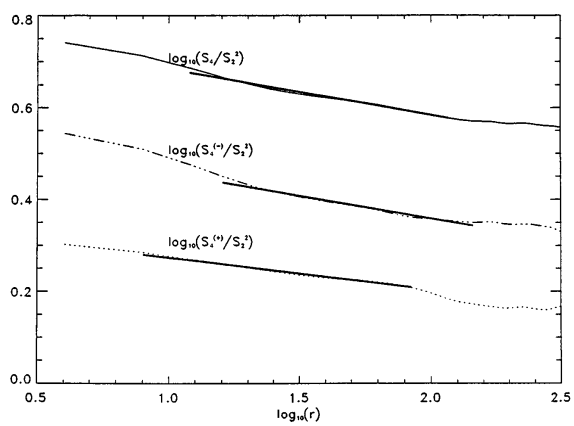


FIG. 4. The logarithm of the ratios S_4/S_2^2, S_4^-/S_2^2, and S_4^+/S_2^2 plotted against $\log r$. These ratios appear to scale, albeit in different ranges of r. From top to bottom, the slopes of the straight line fits and the least square errors are, respectively, -0.099 ± 0.003, -0.098 ± 0.004, and -0.070 ± 0.002. This shows that the plus exponent is larger than the minus exponent.

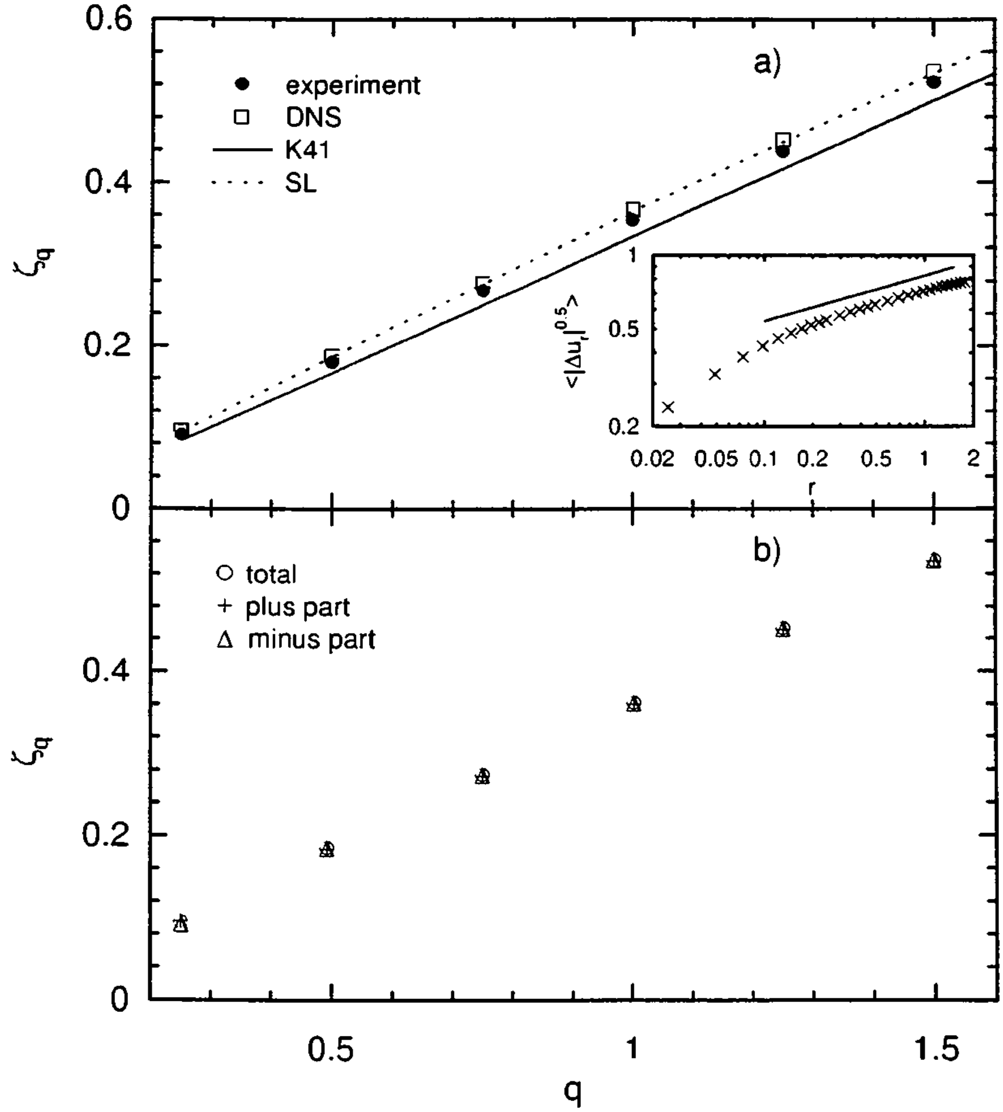


FIG. 5. (a) Low-order scaling exponents for generalized structure functions compared with $q/3$ as well as with outcomes of the intermittency models of [19] and [21]. The inset shows the scaling of the generalized structure function of order 0.5. The full line is for the model of Ref. [21]. (b) Comparison of the plus and minus exponents with those of generalized structure functions.

⋆ S. Chen, et al., “Refined similarity hypothesis for transverse structure functions in fluid turbulence,” Phys. Rev. Lett. **79**, 2253-2256(1997).

This paper is one of several papers around 1997 (see references therein) that studied differences in scaling of longitudinal and transverse velocity structure functions

$$\langle[\delta\mathbf{v}_T(\mathbf{r})]^p\rangle \sim C_p^T \ell^{\zeta_p^T}.$$

The paper reports evidence that

$$\zeta_p^T < \zeta_p^L.$$

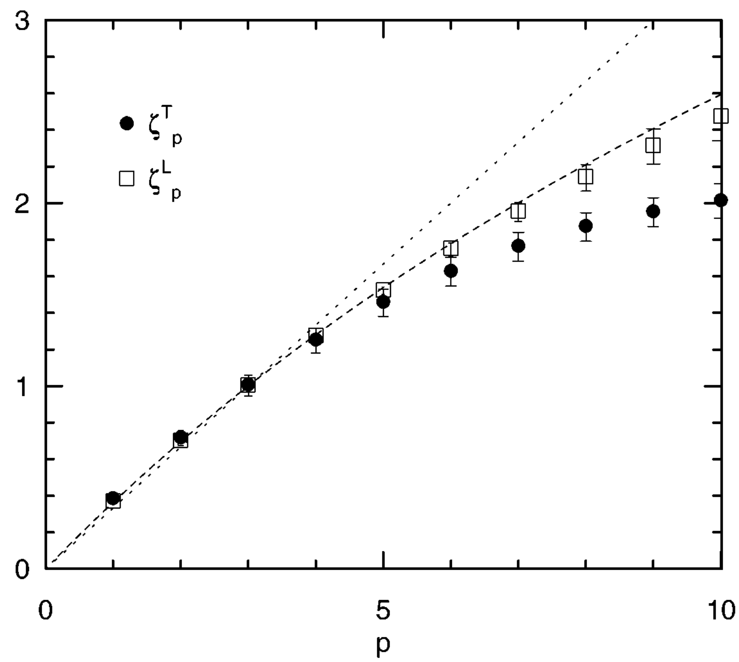


FIG. 3. Numerical results for the transverse scaling exponents, ζ_p^T and ζ_p^L, as functions of p. The dotted line is for the normal scaling relation (K41) and the dashed line is for the log-Poisson model [10].

A more recent systematic study of this issue is contained in

⋆ T. Gotoh et al., "Velocity fields statistics in homogeneous steady turbulence obtained using a high-resolution direct numerical simulation," Phys. Fluids, **14** 1065-1081(2002)

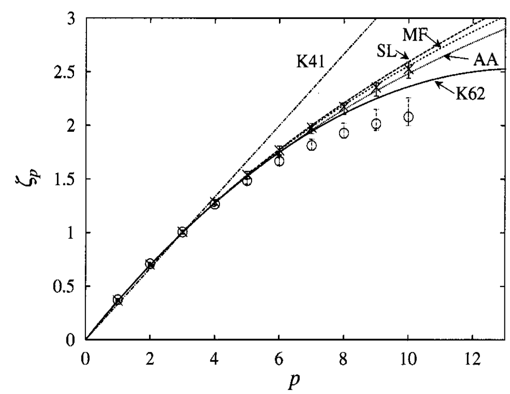


FIG. 32. Variation of the scaling exponents ζ_p^L and ζ_p^T when $R_\lambda = 460$. Symbols are the results of the present DNS, star: ζ_p^L, circle: ζ_p^T. SL, MF, AA, and K62 are the curves by She and Lévêque model (Ref. 70), Yakhot's mean field theory (Ref. 72), Arimitsu and Arimitsu's generalized entropy theory with μ=0.25, and K62 with μ=0.25, respectively.

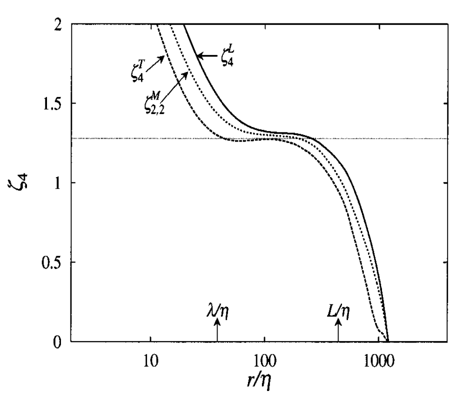


FIG. 33. Comparison of the fourth order $\zeta_{2,2}^M$. $R_\lambda = 381$. A horizontal line indicates 1.28.

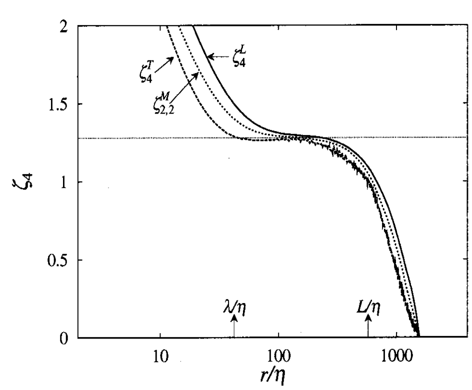


FIG. 34. Comparison of the fourth order scaling exponents ζ_4^L, ζ_4^T, and $\zeta_{2,2}^M$. $R_\lambda = 460$. A horizontal line indicates 1.28.

There has been much phenomenological work to "fit" the scaling exponents with simple analytic formulas. For example,

> Z. S. She and E. Lévêque, "Universal scaling laws in fully developed turbulence," Phys. Rev. Lett. **72** 336-339(1994)

argues that

$$\zeta_p^L = \frac{p}{9} + 2 - 2(\tfrac{2}{3})^{p/3}.$$

This fits the data quite well. (Note in particular that $\zeta_3^L = 1$ as required). There are some interesting physical ideas behind this formula, but no substantial use of the Navier-Stokes dynamics was made in its derivation. For more about early phenomenological models, consult the book of U. Frisch!

More use is made of the Navier-Stokes equation in Lagrangian stochastic models, such as

> L. Chevillard & C. Meneveau, "Intermittency and university in a Lagrangian model of velocity gradients in three-dimensional turbulence," Comptes Rendus Mecanique **335** 187-193 (2007). Arxiv.org: physics/0701274

There is, however, no generally accepted theory of any of the scaling exponents for Navier-Stokes turbulence (except $\zeta_3^L = 1$ of course!) There is still a big open problem.

Successful analytical calculations have been made in toy models, such as Burgers equation. More recently, very important results were obtained in the Kraichnan model of random advection. This work is reviewed in:

> G. Falkovich, K. Gawędzki & M. Vergassola, "Particles and fields in fluid turbulence," Rev. Mod. Phys. **73** 913-975 (2001)

In the Kraichnan model successful calculations have been made of scaling exponents ζ_n for scalar structure functions.

ALL of this work, however, requires the statistical approach, which is outside the scope of this course.

(E) The Multifractal Model

We shall now return to an important general explanation/picture of anomalous scaling, the multifractal model. This framework does not provide a quantitative prediction for scaling exponents, but it does yield some predictions about the relation of the ζ_p's to other measurable quantities, as well as some useful insights.

Local Hölder Exponents & Fractal Dimensions

We have defined earlier the notion of Hölder continuity with an exponent h at point $\mathbf{x}$. However, we have not defined the notion of the Hölder exponent $h(\mathbf{x})$. Ideally, this should mean something like

$$|\delta\mathbf{v}(\mathbf{r},\mathbf{x})| \sim v_{rms}(\tfrac{\ell}{L})^h \quad \text{, with} \quad h = h(\mathbf{x}) \quad \text{when} \quad \ell \ll L,$$

or, more precisely, that

$$\lim_{r\to 0} \tfrac{\ln|\delta\mathbf{v}(\mathbf{r},\mathbf{x})|}{\ln(r/L)} = h \equiv h(\mathbf{x})$$

However, this limit need not exist! Thus, we must follow a similar strategy as far as for the global scaling exponent ζ_p and define

$$h(\mathbf{x}) = \liminf_{r\to 0} \tfrac{\ln|\delta\mathbf{v}(\mathbf{r};\mathbf{x})|}{\ln(r/L)} = \lim_{\ell\to 0} \inf_{|\mathbf{r}|<\ell} \tfrac{\ln|\delta\mathbf{v}(\mathbf{r};\mathbf{x})|}{\ln(r/L)}.$$

This is the maximal Hölder exponent of $\mathbf{v}$ at $\mathbf{x}$, i.e. $\mathbf{v}$ is Hölder continuous with exponent $h = h(\mathbf{x}) - \varepsilon$ at $\mathbf{x}$ for any $\varepsilon > 0$ but *not* with exponent $h(\mathbf{x}) + \varepsilon$ for any $\epsilon > 0$. Just as for the global scaling exponents ζ_p, we employ definitions like those above under the assumption that viscosity $\nu \to 0$ first (before $r \to 0$). Otherwise, we are interested in the value of h defined by an intermediate asymptotics for some range of length-scales $\eta_h(\mathbf{x}) \ll \ell \ll L$. If $\delta u(\mathbf{r};\mathbf{x}) \simeq (\mathbf{r}\cdot\boldsymbol{\nabla})\mathbf{v}(\mathbf{x})$ for $r \lesssim \eta_h(\mathbf{x})$, then

$$\tfrac{\ln|\delta\mathbf{v}(\mathbf{r};\mathbf{x})|}{\ln(r/L)} \simeq 1 \quad \text{for} \quad r \lesssim \eta_h(\mathbf{x}),$$

and the infimum over $|\mathbf{r}| < \ell$ is achieved for $r \gtrsim \eta_h(\mathbf{x})$. A little later we shall present some heuristic estimates of the size of $\eta_h(\mathbf{x})$.

The above definitions are adequate for $0 < h < 1$, but not otherwise. For $h > 1$, we say that $\mathbf{v}$ is Hölder continuous at $\mathbf{x}$ with exponent h if $\mathbf{v}$ is k-times continuously differentiable

$$k = [\![h]\!] = \text{ integer part of } h$$

and if

$$\boldsymbol{\nabla}^k \mathbf{v} \in C^s(\mathbf{x}),\ s = (\!(h)\!) = h - [\![h]\!] = \text{ fractional part of } h.$$

But, for this definition, if $h > 1$, then

$$\begin{aligned} \delta\mathbf{v}(\mathbf{r};\mathbf{x}) &= (\mathbf{r}\cdot\boldsymbol{\nabla})\mathbf{v}(\mathbf{x}_*),\ \mathbf{x}_* \text{ between } \mathbf{x} \text{ and } \mathbf{x}+\mathbf{r} \\ \Longrightarrow \delta u(\ell;\mathbf{x}) &= O(\ell) \end{aligned}$$

Hence, (first-order) increments do not distinguish different values of h! There are various alternatives that allow one to define Hölder exponents for h outside the range $0 < h < 1$.

One alternative is to consider higher-order increments $\delta^k\mathbf{v}(\mathbf{r})$, defined iteratively by

$$\begin{aligned} \delta^1\mathbf{v}(\mathbf{r};\mathbf{x}) &= \delta\mathbf{v}(\mathbf{r};\mathbf{x}) = \mathbf{v}(\mathbf{x}+\mathbf{r}) - \mathbf{v}(\mathbf{x}) \\ \delta^k\mathbf{v}(\mathbf{r};\mathbf{x}) &= \delta^{k-1}\mathbf{v}(\mathbf{r};\mathbf{x}+\mathbf{r}) - \delta^{k-1}\mathbf{v}(\mathbf{r};\mathbf{x}),\quad k \geq 2 \end{aligned}$$

For example,

$$\delta^2\mathbf{v}(\mathbf{r};\mathbf{x}) = \mathbf{v}(\mathbf{x}+2\mathbf{r}) + \mathbf{v}(\mathbf{x}) - 2\mathbf{v}(\mathbf{x}+\mathbf{r}),$$

or, perhaps the more symmetrical choice,

$$\delta_-\delta_+\mathbf{v}(\mathbf{r};\mathbf{x}) = \mathbf{v}(\mathbf{x}+\mathbf{r}) + \mathbf{v}(\mathbf{x}-\mathbf{r}) - 2\mathbf{v}(\mathbf{x}),$$

by a combination of a forward difference $\delta_+\mathbf{v}(\mathbf{r};\mathbf{x}) = \mathbf{v}(\mathbf{x}+\mathbf{r}) - \mathbf{v}(\mathbf{x})$ and a backward difference $\delta_-\mathbf{v}(\mathbf{r};\mathbf{x}) = \mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{x}-\mathbf{r})$. It then turns out that, for all $h > 0$,

$$\mathbf{v} \in C^h(\mathbf{x}) \Longleftrightarrow \delta^k\mathbf{v}(\mathbf{r};\mathbf{x}) = O(\ell^h) \quad \text{for} \quad k \geq [\![h]\!].$$

Another possibility is to use wavelet coefficients. Suppose that ψ_ν for $\nu = 1, \cdots, 2^d - 1$, are "mother wavelets" and ϕ an associated "scale function" which generate an orthonormal basis in $L^2(\mathbb{R}^d)$, as

$$\psi_{N\mathbf{n}\nu}(\mathbf{x}) = 2^{dN/2}\psi_\nu(2^N\mathbf{x} - \mathbf{n})$$

and

$$\phi_{0,\mathbf{n}}(\mathbf{x}) = \phi(\mathbf{x}-\mathbf{n}),$$

$$\begin{aligned} N &\in \mathbb{N}, \text{ the scale index} \\ \mathbf{n} &\in \mathbb{Z}^d, \text{ the position index} \\ \nu &= 1, \cdots, 2^d-1, \text{ the degeneracy index} \end{aligned}$$

We may then expand each component of $\mathbf{v}$ as

$$v_i(\mathbf{x}) = \sum_{\mathbf{n}} B_{\mathbf{n}}^{(i)} \phi_{0,\mathbf{n}}(\mathbf{x}) + \sum_{N\mathbf{n}\nu} A_{N\mathbf{n}\nu}^{(i)} \psi_{N\mathbf{n}\nu}(\mathbf{x})$$

with

$$B_{\mathbf{n}}^{(i)} =< v_i, \phi_{0,\mathbf{n}} >, A_{N\mathbf{n}\nu}^{(i)} =< v_i, \psi_{N\mathbf{n}\nu} >$$

the wavelet coefficients of v_i. It is known that the wavelet coefficients give "almost" a characterization of $C^h(\mathbf{x})$. That is,

$$\mathbf{v} \in C^h(\mathbf{x}) \Longrightarrow \sup_{\{i,\mathbf{n},\nu:\, \mathrm{supp}(\psi_{N\mathbf{n}\nu})\ni\mathbf{x}\}} |A_{N\mathbf{n}\nu}^{(i)}| = O(2^{-N(\frac{d}{2}+h)}), \quad \forall N \geq 1$$

and, conversely,

$$\begin{aligned} \sup_{\{i,\mathbf{n},\nu:\, \mathrm{supp}(\psi_{N\mathbf{n}\nu})\ni\mathbf{x}\}} |A_{N\mathbf{n}\nu}^{(i)}| &= O(2^{-N(\frac{d}{2}+h)}), \quad \forall N \geq 1 \\ \Longrightarrow \delta\mathbf{v}(\mathbf{r};\mathbf{x}) &= O(|\mathbf{r}|^h \ln(\frac{2}{|\mathbf{r}|})). \end{aligned}$$

Here it is assumed that $0 < h < 1$ and that ψ_ν are C^1 and compactly supported. The similar statement applies for any $h > 0$, with $\delta\mathbf{v} \to \delta^k\mathbf{v}$ for $k = [\![s]\!]$, if one assumes also that $\psi_\nu \in C^{k+1}$, such that ψ_ν has all moments of order less than $k+1$ vanishing, i.e.

$$\int d^d\mathbf{x}\ P_k(\mathbf{x})\psi_\nu(\mathbf{x}) = 0$$

for any polynomial $P_k(\mathbf{x})$ of degree $\leq k$. One can therefore also define

$$h(\mathbf{x}) = \liminf_{N\to\infty} \frac{\sup_{\{i,\mathbf{n},\nu:\mathrm{supp}(\psi_{N\mathbf{n}\nu})\ni\mathbf{x}\}} \log_2[2^{Nd/2}|A_{N\mathbf{n}\nu}^{(i)}|]}{-N}$$

This definition has advantages because it also extends to negative Hölder singularities. One could intuitively expect that a power-law blow-up, like

$$|\mathbf{v}(\mathbf{x}+\mathbf{r})| \sim (r/L)^h \quad \text{for} \quad r \ll L \quad \text{with} \quad h < 0$$

would correspond to $\mathbf{v} \in C^h(\mathbf{x})$, i.e. $\mathbf{v}$ belonging to a negative Hölder class at $\mathbf{x}$. It can be shown that this is true if one defines $C^h(\mathbf{x})$ to consist of $\mathbf{v}$ such that

$$\sup_{\{i,\mathbf{n},\nu:\ \mathrm{supp}(\psi_{N\mathbf{n}\nu})\ni\mathbf{x}\}} |A^{(i)}_{N\mathbf{n}\nu}| = O(2^{-N(\frac{d}{2}+h)})$$

for $h > -d$. This doesn't exactly coincide with the standard definition of Hölder continuity for $h > 0$, but the maximal Hölder exponents $h(\mathbf{x})$ are the same.

References

For characterization of Hölder spaces (and, more generally, Besov spaces) by differences:

H. Triebel, *Theory of Function Spaces* (Birkhäuser, 1983), Section 2.5.10-12.

For characterization of Hölder spaces (and, more generally, Besov spaces) by wavelet coefficients:

I. Daubechies, *Ten Lectures on Wavelets* (SIAM, Philadelphia, 1992), Chapter 9.

M. Frazier, B. Jawerth & G. Weiss, *Littlewood-Paley Theory and the Study of Function Spaces* (American Mathematical Society, Providence, RI, 1991), Ch. 7.

For a discussion of negative Hölder singularities, defined by means of wavelet coefficients:

S. Jaffard, "Multifractal formulation for functions, Parts I, II," SIAM J. Math. Annal., **28**(4) 944-970, 971-998(1997).

G. L. Eyink, "Besov spaces and the multifractal hypothesis," J. Stat. Phys. **78** 353-375 (1995).

Last remark: One can similarly define (assuming $0 < h < 1$)

$$\begin{aligned} h_L(\mathbf{x}) &= \liminf_{r\to 0} \frac{\ln|\delta v_L(\mathbf{r};\mathbf{x})|}{\ln(r/L)} = \text{longitudinal Hölder exponent of } \mathbf{v} \text{ at point } \mathbf{x} \\ h_T(\mathbf{x}) &= \liminf_{r\to 0} \frac{\ln|\delta v_T(\mathbf{r};\mathbf{x})|}{\ln(r/L)} = \text{transverse Hölder exponent of } \mathbf{v} \text{ at point } \mathbf{x} \end{aligned}$$

Since $|\delta\mathbf{v}(\mathbf{r})|^2 = |\delta v_L(\mathbf{r})|^2 + (d-1)|\delta v_T(\mathbf{r})|^2$, it is easy to see that $h(\mathbf{x}) = \min\{h_L(\mathbf{x}), h_T(\mathbf{x})\}$.

Fractal Dimensions

We now consider some of the notions of non-integer dimensionality for sets $S \subset \mathbb{R}^d$. We first consider the so-called box-counting dimension $D_B(S)$ (also known as packing dimension, Minkowski-Bouligand dimension, or simply fractal dimension). If $S \subset \mathbb{R}^d$ is bounded, let

$$N_k(S) = \text{ numbers of hypercubes in a regular grid of sidelength } 2^{-k} \text{ which intersect } S$$

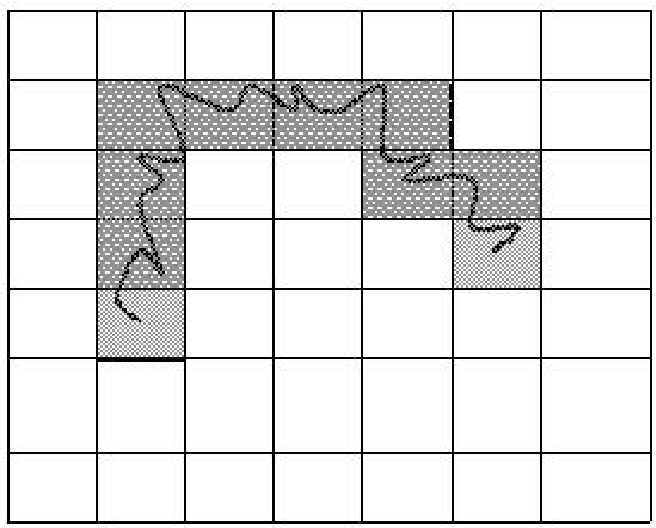


Figure 1. Illustration of the box-counting method

Define

$$\begin{aligned} \overline{D}_B(S) &= \limsup_{k\to\infty} \frac{1}{k} \log_2 N_k(S) = \text{ upper fractal dimension} \\ \underline{D}_B(S) &= \liminf_{k\to\infty} \frac{1}{k} \log_2 N_k(S) = \text{ lower fractal dimension} \end{aligned}$$

so that $\quad \underline{D}_B(S) \leq \overline{D}_B(S)$.

If $\overline{D}_B(S) = \underline{D}_B(S) = D_B$, then we say that S has fractal dimension D_B and, in that case,

$$N_k(S) \sim 2^{kD_B}.$$

Notice that the total number of boxes per unit volume grows as

$$N_k \sim 2^{kd}.$$

Hence, the fraction of boxes intersecting S goes as

$$\tfrac{N_k(S)}{N_k} \sim 2^{-k(d-D_B(S))} \sim 2^{-k\cdot\kappa_B(S)}$$

where

$$\kappa_B(S) = d - D_B(S) = \text{fractal codimension of } S.$$

Remark: Rather than boxes of sidelength 2^{-k}, decreasing by factors of 2, one can use any factor $\lambda > 1$ and sidelengths λ^{-k}. In that case, $N_k(S) \sim \lambda^{kD_B(S)}$ with the same value $D_B(S)$, independent of the λ selected.

Example: Consider the middle-thirds Cantor set $K = \cap K_n$:

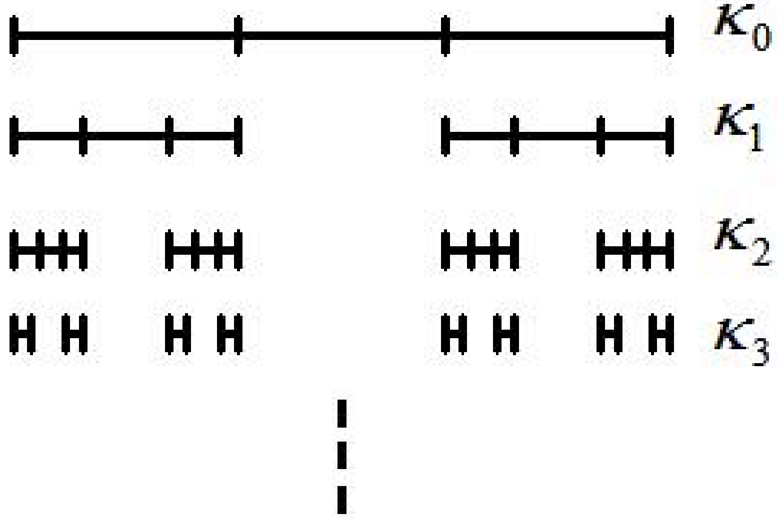


Figure 1. Iterative construction of the Cantor set

Choosing $\lambda = 3$, we get (by direct counting)

$$N_k(K) = 2^k$$

so that

$$\begin{aligned} D_B(K) &= \lim_{k\to\infty} \frac{\log_3 N_k(K)}{k} \\ &= \lim_{k\to\infty} \frac{\log_3(2^k)}{k} \\ &= \lim_{k\to\infty} \frac{k \cdot \log_3(2)}{k} = \log_3(2) = \frac{\ln 2}{\ln 3} \doteq 0.6309\cdots \end{aligned}$$

There are many other important notions of fractal dimensionality, e.g. Hausdorff dimension $D_H(S)$. A full discussion of $D_H(S)$ requires measure theory, but, essentially, the difference is that it involves covering the set with balls of size $\leq 2^{-k}$ and not just $= 2^k$, so that the number of balls required is always fewer (and could be much fewer). Thus,

$$D_H(S) \leq \underline{D}_B(S) \leq \overline{D}_B(S).$$

For a complete discussion of all these matters, see

> K. Falconer, *Fractal Geometry, Mathematical Foundations and Applications.* (John Wiley & Sons, 1990)

<u>Parisi-Frisch Multifractal Model</u>

We now consider the heuristic explanation given by

> G. Parisi & U. Frisch, "On the singularity structure of fully developed turbulence," in: *Turbulence and Predictability in Geophysical Fluid Dynamics.* Proc. Internatl. School of Physics 'E. Fermi', 1983, Varenna, Italy, 84-87, eds. M. Ghil, R. Benzi and G. Parisi. (North-Holland, Amstordam, 1985)

for the anomalous scaling of structure functions. We assume that the local Hölder exponents $h(\mathbf{x})$ of the velocity $\mathbf{v}$ obtained for $\nu \to 0$ all lie in some range $[h_{min}, h_{max}]$ with $0 < h_{min}$ and $h_{max} < 1$. We then set

$$\mathcal{S}(h) = \{\mathbf{x} : h(\mathbf{x}) = h\}$$

and

$$D(h) = D(\mathcal{S}(h)).$$

Since the argument is just heuristic, we do not state here whether we specify $D = D_B, D_H$ or some other fractal dimension! Now, by definition of $\mathcal{S}(h)$,

$$|\delta\mathbf{v}(\mathbf{r};\mathbf{x})| \sim v_0(\tfrac{r}{L})^h$$

if $\mathbf{x}$ is within distance $\mathbf{r}$ of the set $\mathcal{S}(h)$. The fraction of space on which that occurs can be estimated as

$$\text{Fraction}(\mathbf{x} : \text{dist}(\mathbf{x}, \mathcal{S}(h)) \sim r) \sim (\tfrac{r}{L})^{\kappa(h)}$$

where $\kappa(h) = d - D(h)$ is the fractal codimension of $\mathcal{S}(h)$. If the distribution of exponents over space has any smooth weight μ, then

$$\begin{aligned} S_p(\mathbf{r}) &= \langle|\delta\mathbf{v}(\mathbf{r})|^p\rangle \\ &\sim \int_{h_{min}}^{h_{max}} d\mu(h)\ \left[v_0(\frac{r}{L})^h\right]^p (\frac{r}{L})^{\kappa(h)} \\ &\sim u_0^p \int_{h_{min}}^{h_{max}} d\mu(h)\ (\frac{r}{L})^{ph+\kappa(h)} \\ &\sim u_0^p (\frac{r}{L})^{\zeta_p} \end{aligned}$$

by steepest descent, or, more precisely,

$$\lim_{r\to 0} \frac{\ln S_p(r)}{\ln(r/L)} = \zeta_p$$

with

$$\boxed{\zeta_p = \inf_{h\in[h_{min},h_{max}]} \{ph + \kappa(h)\}}$$

<u>Example: Burgers equation</u>

There are two Hölder exponents

$$\text{isolated shocks,}\ \ h = 0 \ \ \text{with} \ \ D(0) = 0 \Longrightarrow \kappa(0) = 1 - D(0) = 1$$

$$\text{smooth ramps,}\ \ h = 1 \ \ \text{with} \ \ D(1) = 1 \Longrightarrow \kappa(1) = 1 - D(1) = 0$$

For other $h \neq 0, 1, \kappa(h) = +\infty$, formally, so that the infimum is taken only over $h = 0$ or 1:

$$\begin{aligned} \zeta_p &= \inf_{n\in\{0,1\}} [ph + \kappa(h)] \\ &= \inf\{1, p\} \\ &= \begin{cases} p & 0 \leq p < 1 \\ 1 & p \geq 1 \end{cases} \end{aligned}$$

The formula works! This is the same answer we obtained before by direct calculation of ζ_p for the Khokhlov sawtooth solution of the Burgers equation. The Burgers solution is an example of a <u>bifractal</u> field, with just two distinct exponents.

Going back to the general theory, the relation between ζ_p and $\kappa(h)$

$$\zeta_p = \inf_h \{ph + \kappa(h)\}$$

is an example of a <u>Legendre transform</u>. E.g. see R. T. Rockafellar, *Convex Analysis* (Princeton University Press, 1970) for more discussion. If $\kappa(h)$ is strictly convex (i.e. $D(h)$ is strictly concave), then the infimum is uniquely achieved for each p at the point $h_*(p)$ where

$$\begin{aligned} 0 = \frac{\partial}{\partial h}\{ph + \kappa(h)\}|_{h=h_*(p)} &= p + \kappa'(h_*(p)) \\ \Longrightarrow \kappa'(h_*(p)) &= -p. \end{aligned}$$

Since $p_*(h) = -\kappa'(h)$ is decreasing in h (by strict convexity of $\kappa(h)$), the inverse function $h_*(p)$ is well-defined and also decreasing in p. Then

$$\zeta_p = \inf_h \{ph + \kappa(h)\} = ph_*(p) + \kappa(h_*(p)).$$

But note that, in that case,

$$\begin{aligned} \frac{d\zeta_p}{dp} &= h_*(p) + \underbrace{[p + \kappa'(h_*(p))]}_{=0} \frac{dh_*(p)}{dp} \\ &= h_*(p) \quad ! \end{aligned}$$

There is a pleasing symmetry that

$$p_*(h) = -\tfrac{d\kappa}{dh}(h) \quad \& \quad h_*(p) = \tfrac{d\zeta_p}{dp}$$

Furthermore,

$$\begin{aligned} \kappa(h_*(p)) &= \zeta_p - ph_*(p) \\ \Longrightarrow \kappa(h) &= \zeta_{p_*(h)} - p_*(h)h \end{aligned}$$

Note that $\zeta_p - ph$ is <u>concave</u> in p and that

$$\tfrac{d}{dp}[\zeta_p - ph] = h_*(p) - h = 0$$

precisely when $h_*(p) = h$ or $p = p_*(h)$. Thus,

$$\kappa(h) = \zeta_{p_*(h)} - p_*(h)h$$

$$= \sup_h \{\zeta_p - ph\} \ !$$

The nice symmetry

$$\zeta_p = \inf_h \{ph + \kappa(h)\} \quad \& \quad \kappa(h) = \sup_h \{\zeta_p - ph\}$$

is an example of Legendre duality. If $\kappa(h)$ is not convex, then it can be shown that $\overline{\kappa}(h) = \sup_p\{\zeta_p - ph\}$ is the convex hull of $\kappa(h)$. See Rockafellar (1970).

It was proposed by Parisi-Frisch (1985) to use this inversion formula to extract the (presumed universal) function $\kappa(h)$ from the data, or, equivalently,

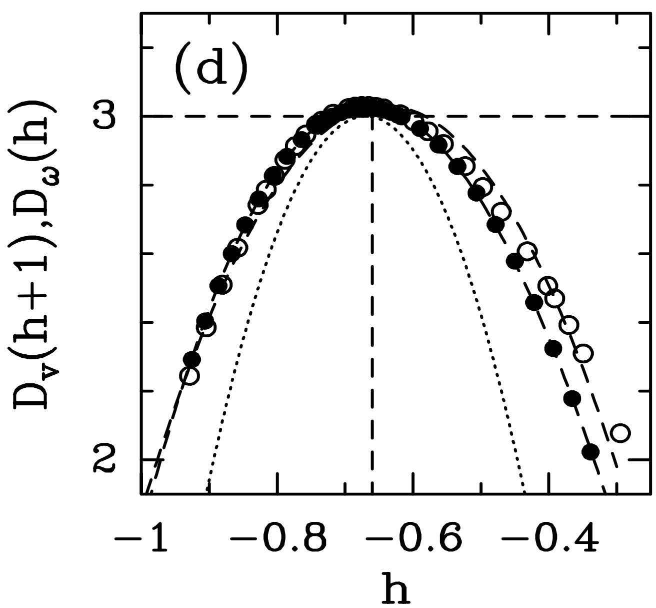
$$D(h) = \inf_p \{ph + (d - \zeta_p)\},$$

the so-called multifractal dimension spectrum. Notice that the domain $I = [h_{min}, h_{max}]$ of the concave function $D(h)$ is just the range of possible slopes of ζ_p, since $h = d\zeta_p/dp$.

This program has been carried out, although the method based on structure functions and their scaling exponents ζ_p has not proved the most accurate or robust method to determine $D(h)$. E.g., a wavelet-transform maximum modulus method has been used instead. See

> P. Kestener & A. Arneodo, "Generalizing the wavelet-based multifractal formulism to random vector fields: Application to three-dimensional turbulence velocity and vorticity data," Phys. Rev. Lett. **93** 044501(2004)

and many references therein. They obtain $D(h)$ spectra of both velocity and vorticity, using data from a 256^3 DNS at $Re_\lambda = 140$. They obtain a most probable $h_* = h_*(0)$ of about 0.34 ± 0.02, just a little bigger than the K41 value $h = 1/3$:



Note, finally, that one can define multifractal spectra of other quantities, such as longitudinal and transverse velocity increments

$$D(h_L) = \dim\{\mathbf{x} : h_L(\mathbf{x}) = h_L\}, \quad D(h_T) = \dim\{\mathbf{x} : h_T(\mathbf{x}) = h_T\},$$

or even joint spectra for both exponents together

$$D(h_L, h_T) = \dim\{\mathbf{x} : h_L(\mathbf{x}) = h_L,\ h_T(\mathbf{x}) = h_T\}.$$

But I'm not aware of any experimental/numerical results on these. We've seen that $p = 0$ corresponds to the "most probable" exponent h_* with

$$D(h_*) = 0 \cdot h_* + (d - \zeta_0) = d.$$

The exponents with $h < h_*$ correspond to $p > 0$, while those with $h > h_*$ correspond to negative $p < 0$. It is possible, but can be quite tricky, to study structure functions of negative order p. An alternative approach based on so-called inverse structure functions is useful here:

M. H. Jensen, "Multiscaling and structure functions in turbulence: an alternative approach," Phys. Rev. Lett. **83** 76-79 (1999)