III Small-Scale Intermittency & Anomalous Scaling

We have seen that turbulent energy dissipation non-vanishing as $Re \to \infty$ requires that

$$\zeta \leq p/3 \quad \text{for} \quad p \geq 3.$$

The K41 theory assumes the "minimal singularity" sufficient to dissipate energy, or $\zeta = p/3$ for all p. However, other possibilities are allowed by the above estimate! In this set of notes we consider the subject of turbulent scaling laws and their relation to turbulent energy cascade.

(A) A Simple Model of Energy Dissipation: Burgers Equation

In this section we consider a simple 1-dimensional PDE model that has non-vanishing energy dissipation for $Re \to \infty$ but for which K41 theory <u>fails</u>. It is a useful counterexample! The model is the <u>1-dimensional Burgers equation</u> for a velocity field $u(x,t)$:

$$\partial_t u + u\partial_x u = \nu \partial_x^2 u.$$

It can also be written as

$$\partial_t u + \partial_x(\tfrac{1}{2}u^2) = \nu \partial_x^2 u$$

so that it corresponds to a conservation of "momentum" $\int u(x,t)dx$. As a simple prototype of turbulence, it was first proposed by J. M. Burgers, " A mathematical model illustrating the theory of turbulence," Adv. Appl. Mech. **1**, 171-199 (1948).

The "energy"

$$E(t) = \tfrac{1}{2}\int u^2(x,t)dx$$

is also conserved in the limit $\nu \to 0$, i.e. for "ideal" Burgers equation, since

$$\partial_t(\tfrac{1}{2}u^2) + \partial_x[\tfrac{1}{3}u^3 - \nu\partial_x(\tfrac{1}{2}u^2)] = -\nu(\partial_x u)^2.$$

Formally, $\nu(\partial_x u)^2 \to 0$ as $\nu \to 0$. However, this is <u>NOT</u> what occurs!

Consider a simple exact solution of 1-D burgers:

$$u(x,t) = \tfrac{1}{t}[x - L\tanh(\tfrac{Lx}{2\nu t})].$$

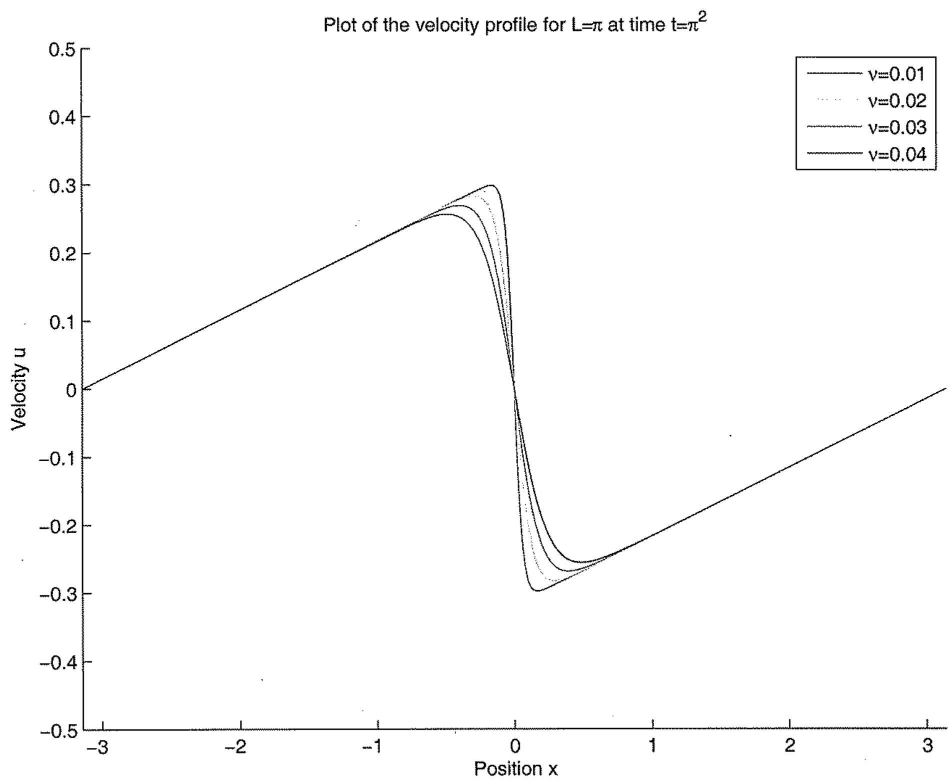
This seems to have been first written down in

S. I. Soluyan & R. V. Khokhlov, Vestnik. Moscow State. Univ. Phys. Astron. **3** 52-61(1961)

and is sometimes called the "Khokhlov saw-tooth solution". The reason for the term "sawtooth" is that

$$\begin{aligned} u^\nu(x,t) &\sim \frac{1}{t}(x+L) \quad \text{as } x \to -\infty \\ u^\nu(x,t) &\sim \frac{1}{t}(x-L) \quad \text{as } x \to +\infty \end{aligned}$$

with a discontinuity $\triangle u = (2L)/t$ across the origin.



Plots of the Khokhlov sawtooth solution at fixed time for various viscosities

In the limit $\nu \to 0$, this becomes

$$u(x,t) = \begin{cases} (x+L)/t & -L \le x < 0 \\ (x-L)/t & 0 < x \le L \end{cases}$$

a function on the interval $[-L, L]$ with $u(\pm L, t) = 0$ a sharp discontinuity of size $\triangle u = (2L)/t$ at $x = 0$. Such a discontinuity is called a shock or, in this case, a stationary shock, since it is located at the same point $x = 0$ for all time t.

Now it is easy to see that there is nonvanishing mean energy dissipation in the limit that $\nu \to 0$. For example, at $\nu = 0$, using the above explicit formula, it is easy to see that

$$\begin{aligned} \frac{1}{2}\langle u^2(t)\rangle &= \frac{1}{2L}\int_{-L}^{L} \frac{1}{2}u^2(x,t)dx \\ &= \frac{1}{2L}\int_{0}^{L} \left(\frac{x-L}{t}\right)^2 dx = \frac{1}{6}\left(\frac{L}{t}\right)^2 \end{aligned}$$

so that

$$\langle \varepsilon(t)\rangle = -\frac{d}{dt}\frac{1}{2}\langle u^2(t)\rangle = \frac{1}{3}\frac{L^2}{t^3} = \frac{(\triangle u)^2}{12t} > 0!!!$$

Alternatively, one can consider the viscous dissipation

$$\varepsilon^\nu(x,t) = \nu|\partial_x u^\nu(x,t)|^2$$

using

$$\partial_x u^\nu(x,t) = \frac{1}{t} - \frac{L^2}{2\nu t^2}\mathrm{sech}^2\left(\frac{Lx}{2\nu t}\right)$$

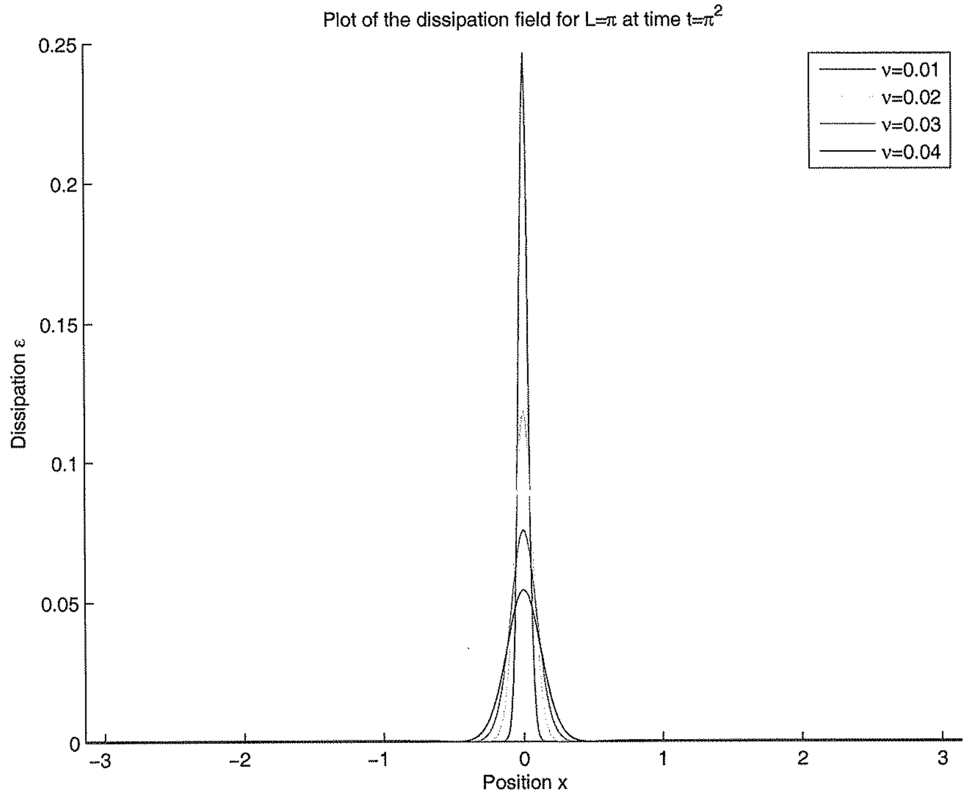
so that $\nu \ll L^2/t$, with L and t fixed,

$$\varepsilon^\nu(x,t) \approx \frac{L^4}{4\nu t^4}\mathrm{sech}^4\left(\frac{Lx}{2\nu t}\right)$$

The energy dissipation becomes very large $\sim \frac{L^4}{\nu t^4}$ in a small region of size $\sim \nu t/L$. Using the simple integral $\int_{-\infty}^{+\infty} \mathrm{sech}^4 u \, du = \frac{4}{3}$, we again finds that

$$\begin{aligned} \langle \varepsilon^\nu(t)\rangle &= \frac{1}{2L}\int_{-L}^{L} \varepsilon^\nu(x,t)dx \\ &\cong \frac{1}{2L}\cdot\frac{L^4}{4\nu t^4}\cdot\frac{2\nu t}{L}\cdot\frac{4}{3} \quad \text{for } \nu \to 0 \\ &= \frac{1}{3}\frac{L^2}{t^3} \quad \text{or} \quad \frac{(\triangle u)^2}{12t} \end{aligned}$$

Again, the limit as $\nu \to 0$ is positive! This is exactly like the experiments & simulations for real fluids!



Plots of energy dissipation in the Khokhlov sawtooth solution for various viscosities

All of our previous theory applies to this problem. E.g. we may consider the coarse-grained equation

$$\partial_t \bar{u}_\ell + \partial_x(\tfrac{1}{2}\bar{u}_\ell^2 + \tau_\ell) = \nu\partial_x^2\bar{u}_\ell$$

with

$$\begin{aligned}\tau_\ell &= \frac{1}{2}[\overline{(u^2)_\ell} - \bar{u}_\ell^2] \\ &= \frac{1}{2}[\langle(\delta u)^2\rangle_\ell - \langle\delta u\rangle_\ell^2] = O(\delta u^2(\ell))\end{aligned}$$

The large-scale energy balance is

$$\partial_t(\tfrac{1}{2}\bar{u}_\ell^2) + \partial_x[\tfrac{1}{3}\bar{u}_\ell^3 + \tau_\ell\bar{u}_\ell - \nu\partial_x(\tfrac{1}{2}\bar{u}_\ell^2)] = \tau_\ell(\partial_x\bar{u}_\ell) - \nu(\partial_x\bar{u}_\ell)^2.$$

For fixed ℓ we see again that $\nu(\partial_x\bar{u}_\ell)^2 = O(\frac{\nu\delta^2 u(\ell)}{\ell^2})$, which goes to zero as $\nu \to 0$. Large-scale "dissipation" must come from

$$\tau_\ell \partial_x \bar{u}_\ell = O(\tfrac{\delta u^3(\ell)}{\ell})$$

from which we can deduce, as before, that

$$\zeta_p \leq p/3 \quad \text{for} \quad p \geq 3$$

However, K41 theory does not work here! For the limiting shock profile at $\nu \to 0$:

$$u(x,t) = \begin{cases} (x+L)/t & -L \leq x < 0 \\ (x-L)/t & 0 < x \leq L \end{cases},$$

periodically extended to $\mathbb{R}$ with period $2L$, we can see that (with $\ell > 0$)

$$u(x+\ell,t) - u(x,t) = \begin{cases} \ell/t & \text{if} \quad 0 \notin [x, x+\ell] \\ (2L+\ell)/t & \text{if} \quad 0 \in [x, x+\ell] \end{cases}$$

$\Longrightarrow$

$$\begin{aligned} \langle |\delta u(\ell)|^p \rangle &= \frac{1}{2L}\int_{-L}^{L} |u(x+\ell) - u(x)|^p dx \\ &= (1 - \frac{\ell}{2L}) \cdot (\frac{\ell}{t})^p + \frac{\ell}{2L} \cdot (\frac{2L+\ell}{t})^p \\ &\sim (\triangle u)^p \begin{cases} (\frac{\ell}{2L})^p & 0 < p < 1 \\ \frac{\ell}{2L} & p \geq 1 \end{cases} \quad \text{for} \ \ \ell \ll L \end{aligned}$$

Thus,

$$\zeta_p = \begin{cases} p & 0 < p < 1 \\ 1 & p \geq 1 \end{cases}$$

Of course, $1 \leq p/3$ for $p \geq 3$, so that our inequality is verified that $\zeta_p \leq p/3$ but only for $p = 3$ is $\zeta_p = p/3$!

The problem is that K41 theory assumes that $h = \frac{1}{3}$ at <u>every point of space</u> and that is not what happens here. Instead, there is <u>one point</u> ($x = 0$) where

$$\delta u(\ell, x) \sim \triangle u \sim \ell^0 \ \ \text{for } \underline{\text{all}} \ \ \ell < L$$

and, at every other point,

$$\delta u(\ell, x) \sim \ell/t \sim \ell^1$$

for sufficiently small ℓ. This is an extreme example of <u>small-scale intermittency</u>, in which velocity increments are "spotty" in space, big in some places and small in others.

Incidentally, it is known that the above features that we have seen in a simple specific solution of Burgers equation are, in fact, generic for that dynamics. Except for fields in which $\partial_x u > 0$ everywhere, the Burgers solutions always develop shocks that become exact discontinuities in the limit $\nu \to 0$ and these dissipate a finite amount of energy that does not vanish as $\nu \to 0$. The scaling exponents ζ_p that we determined above are also universal to a wide class of initial data and forcing schemes.

For more information about Burgers equation and "Burgulence," see :

J. M. Burgers, The Non-Linear Diffusion Equation: Asymptotic Solutions & Statistical Problems, (Springer, Boston, 1974)

W. E. K. Khanin, A. Mazel and Y. Sinai, "Invariant Measures for Burgers Equation with Stochastic Forcing," Ann. Math. **151** (3): 817-960 (2000)

and a review article

J. Bec & K. Khanin, "Burgers Turbulence," Physics Reports **447** 1-66 (2007)

Before we leave the topic of Burgers equation, there is one last important remark we wish to make. Consider again the limiting shock profile

$$u(x,t) = \begin{cases} (x+L)/t & -L \leq x < 0 \\ (x-L)/t & 0 < x \leq L \end{cases} \quad \text{periodic on } [-L, L]$$

These results are just special cases of a general theory for Burgers equation (and for a whole class of scalar conservation laws in 1-dimension). It is known that the solution $u^\nu(x,t)$ of

$$\partial_t u^\nu(x,t) + u^\nu(x,t)\partial_x u^\nu(x,t) = \nu \partial_x^2 u^\nu(x,t)$$

converges in the sense of distributions $u^\nu(x,t) \to u(x,t)$ as $\nu \to 0$ to a solution of the inviscid equation

$$\partial_t u(x,t) + \partial_x(\tfrac{1}{2}u^2(x,t)) = 0$$

which satisfies

$$\partial_t(\tfrac{1}{2}u^2(x,t)) + \partial_x(\tfrac{1}{3}u^3(x,t)) \leq 0,$$

both in the sense of distributions. It is quite easy to check that this profile satisfies the inviscid Burgers equation

$$\partial_t u(x,t) + u(x,t)\partial_x u(x,t) = 0$$

at every spacetime point (x,t) with $x \neq 0$. The problem at $x = 0$ is that a derivative $(\partial_x u)$ does not even exist in the classical sense. However, it is not hard to check that the limiting shock profile satisfies everywhere in space and for all $\ell > 0$ the coarse-grained equations.

$$\partial_t \bar{u}_\ell + \partial_x \overline{(\tfrac{1}{2}u^2)}_\ell = 0 \tag{*}$$

or, with a mere change of notation,

$$\partial_t \bar{u}_\ell + \partial_x (\tfrac{1}{2}\bar{u}_\ell^2 + \tau_\ell) = 0$$

with $\tau_\ell = \frac{1}{2}[\overline{(u^2)}_\ell - \bar{u}_\ell^2]$, where all derivatives of the coarse-grained variables do exist in a classical sense. Here the initial data $\bar{u}_\ell(x,0)$ for (*) are also coarse-grained. An equivalent, more conventional formulation of (*) is obtained by noting that derivatives can be taken to exist in the sense of distributions and it can easily be verified that

$$\int_0^\infty dt \int_{-L}^L dx [u(x,t)\partial_t \varphi(x,t) + \tfrac{1}{2}u^2(x,t)\partial_x \varphi(x,t)] + \int_{-L}^L dx\; u(x,0)\varphi(x,0) = 0$$

for all C^∞ functions φ on $[-L,L] \times [0,\infty]$ with compact support (in time). Thus, $u(x,t)$ satifies the inviscid Burgers equation in the sense of distributions, with initial data $u(x,0)$. However, it is also easy to check that energy is not conserved and, in fact, that

$$\partial_t(\tfrac{1}{2}u^2(x,t)) + \partial_x(\tfrac{1}{3}u^3(x,t)) = -\tfrac{(\Delta u)^3}{12}\delta(x) \leq 0 \tag{**}$$

in the sense of distributions.

Furthermore, for a given initial datum $u(x,0)$ this distributional solution is unique. Note that, in general, such "weak" or distributional solutions are not unique and there can be more than one solution even for the same initial data. See examples in

P. D. Lax, "Hyperbolic systems of conservation laws, II." Commun. Pure Appl. Math. **10** 537-566 (1957)

It does not make sense to talk about *the* solution of the inviscid equation — it is an ill-defined concept! It is only weak solutions with the energy inequality (**) that are unique. It is expected that any hyperviscous regularized solution

$$\partial_t u^\epsilon(x,t) + \partial_x(\tfrac{1}{2}|u^\epsilon(x,t)|^2) = -\epsilon(-\partial_x^2)^p u^\epsilon(x,t), \quad \epsilon > 0$$

with $p \geq 1$ converges as $\epsilon \to 0$ to the same $u(x,t)$ as for $p = 1$. See:

E. Tadmor, "Burgers equation with vanishing hyperviscosity," Comm. Math.Sci. **2**(2), 317-324(2004).

However, a dispersive regularization such as the famous Korteweg-de Vries (KDV) equation

$$\partial_t u^\epsilon(x,t) + \partial_x(\tfrac{1}{2}|u^\epsilon(x,t)|^2) = \epsilon\partial_x^3 u^\epsilon(x,t), \quad \epsilon > 0$$

which describes weakly nonlinear shallow water waves, has a completely different class of solutions, even for the same initial data $u(x,0)$. See

P. D. Lax and C. D. Levermore, "The small dispersion limit for the KDV equation, I-III," Comm. Pure Appl. Math. **36** 253-290; 571-594; 889-829 (1983).

For more discussion of these matters see, e.g.

P. D. Lax, "Shock waves and entropy," in: Contributions to Nonlinear Functional Analysis, ed. E.H. Zarantonello (Academic Press, NY, 1971)

R. J. DiPerna, "Measure-valued solutions to conservation laws," Arch. Rat. Mech. Anal., **8** 223-270(1985).

L. C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, vol. 19 (American Mathematical Society, Providence, RI, 1998)

The issues are especially interesting since it was conjectured by Lars Onsager in 1949 that real turbulent fluids are described in the limit $\nu \to 0$ (or $Re \to \infty$) by singular solutions of 3D incompressible Euler equtions that dissipate energy! See

L. Onsager, "Statistical hydrodynamics," Nuovo Cimento, **6** 279-287 (1949)

We shall discuss this in more detail a bit later...

(B) Scaling & Scaling Exponents

See UF, Section 8.4

We have been discussing scaling exponents σ_p

$$r \leq L, \quad \|\delta \mathbf{v}(\mathbf{r})\|_p \sim u_{rms}(r/L)^{\sigma_p}, \quad (\star)$$

or equivalently, $\zeta_p = p\sigma_p$

$$r \leq L, \quad S_p(\mathbf{r}) \sim u_{rms}^p (r/L)^{\zeta_p}, \quad (\star\star)$$

with $S_p(\mathbf{r}) = \langle |\delta \mathbf{v}(\mathbf{r})|^p \rangle = \|\delta \mathbf{v}(\mathbf{r})\|_p^p$, the pth-order structure functions. (To be precise, this is the pth-order absolute structure function; see below). However, we have not yet given a precise definition of σ_p or ζ_p. In particular, there is no proof from first principles that scaling laws like $(\star)$ or $(\star\star)$ even hold! We would like to make definitions of σ_p and ζ_p that are guaranteed to exist, at least when $\mathbf{v} \in L_p$.

A precise statement of $(\star)$ is that

$$\lim_{r \to 0} \frac{\ln \|\delta \mathbf{v}(\mathbf{r})\|_p}{\ln(r/L)} = \sigma_p$$

This means that a plot of $\ln \|\delta \mathbf{v}(\mathbf{r})\|_p$ vs. $\ln(r/L)$ should better approximate a straight line for $r \leq L$ and getting smaller. Unfortunately, this limit is not guaranteed to exist! Instead, we can consider

$$\inf_{r < \ell} \frac{\ln \|\delta \mathbf{v}(\mathbf{r})\|_p}{\ln(r/L)}$$

which is a function of ℓ that is increasing. Thus, its limit as $\ell \to 0$ is guaranteed to exist (possibly $= +\infty$), a quantity which in mathematics is called the limit-infimum and denoted by $\lim_{r\to 0} \inf$. We therefore define

$$\sigma_p \equiv \lim_{r \to 0} \inf \frac{\ln \|\delta \mathbf{v}(\mathbf{r})\|_p}{\ln(r/L)}.$$

[By the way, this is properly called the lower exponent of order p and denoted by $\underline{\sigma}_p$. There is also an upper exponent $\bar{\sigma}_p$ obtained by replacing "inf" with "sup". This limit also exists, because now the function of ℓ is decreasing, and is called the limit-supremum. However, for reasons discussed just below, we refer to use the limit infimum.]

What does this definition mean? Unwrapping it, we see that $\forall \epsilon > 0$, $\exists \delta > 0$, such that

$$\ell < \delta \Longrightarrow \sigma_p - \epsilon \leq \inf_{r<\ell} \frac{\ln \|\delta \mathbf{v}(\mathbf{r})\|_p}{\ln(r/L)} \leq \sigma_p + \epsilon$$

In other words, for all $\ell < \delta$,

$$\exists r \leq \ell, \frac{\ln \|\delta \mathbf{v}(\mathbf{r})\|_p}{\ln(r/L)} \leq \sigma_p + \epsilon \text{ and } \forall r \leq \ell, \frac{\ln \|\delta \mathbf{v}(\mathbf{r})\|_p}{\ln(r/L)} \geq \sigma_p - \epsilon$$

or, again, for all $\ell < \delta$

$$\exists r \leq \ell, \|\delta \mathbf{v}(\mathbf{r})\|_p \geq (\tfrac{r}{L})^{\sigma_p+\epsilon} \text{ and } \forall r \leq \ell, \|\delta \mathbf{v}(\mathbf{r})\|_p \leq (\tfrac{r}{L})^{\sigma_p-\epsilon}$$

Note that $\ln(r/L) < 0$, since $r < L$. Now, the second inequality above means that $\mathbf{v}$ is Besov regular of order p with exponent $\sigma_p - \epsilon$, or that

$$\mathbf{v} \in B_p^{\zeta_p-\epsilon,\infty} = \begin{array}{l} \text{space of functions in } L_p \\ \text{with } \|\delta \mathbf{v}(\mathbf{r})\|_p \leq (const.)(r/L)^{\sigma_p-\epsilon} \\ \text{for some constant.} \end{array}$$

On the other hand, the first inequality above means that $\mathbf{v}$ is NOT Besov regular of order p with exponent $\sigma_p + 2\epsilon$ (for example). Thus, we see that, because $\epsilon > 0$ was arbitrary,

$$\sigma_p \equiv \lim_{r\to 0} \inf \frac{\ln \|\delta \mathbf{v}(\mathbf{r})\|_p}{\ln(r/L)} = \text{maximal Besov exponent of order } p.$$

In other words, $\mathbf{v} \in B_p^{\sigma,\infty}$ for any $\sigma < \sigma_p$ and $\mathbf{v} \notin B_p^{\sigma,\infty}$ for any $\sigma > \sigma_p$. For simplicity, this is sometimes called just the Besov exponent of $\mathbf{v}$ of order p, since it is uniquely defined whenever $\mathbf{v} \in L_p$.

In the same manner, we can define

$$\zeta_p = \lim_{r\to 0} \inf \frac{\ln S_p(\mathbf{r})}{\ln(r/L)}$$

so that $\zeta_p = p\sigma_p$. This definition is satisfactory if we take the limit $\nu \to 0$ first (assuming that limit exists!) In that case, an infinitely long inertial range exists with singular velocity fields. However, for finite $\nu > 0, \sigma_p = 1$ and $\zeta_p = p$ for all p if — as seems plausible for the

Navier-Stokes solution — the velocity is smooth[1] . In that case, $\|\delta\mathbf{v}(\mathbf{r})\|_p \sim (const.)(r/L)$ for sufficiently small r.

Instead, one can hope to get scaling with $\sigma_p < 1$ or $\zeta_p < p$ only in an intermediate asymptotic sense. That is, there should exist a small length-scale

$$\eta_p = (const.)L(Re)^{-\alpha_p}, \ \ \alpha_p > 0$$

such that

$$\inf_{r<\ell} \frac{\ln \|\delta\mathbf{v}(\mathbf{r})\|_p}{\ln(r/L)} \rightarrow \sigma_p$$

better and better for decreasing ℓ in the range $\eta_p \leq \ell \leq L$. Here it is key that this intermediate range should be able to be made longer and longer as $Re \rightarrow \infty$.

Defining ζ_p and σ_p as above, either by taking the limit $Re \rightarrow \infty$ first or else by intermediate asymptotics for $Re \gg 1$, we can then establish some important general properties:

Proposition 1: ζ_p is a concave function of $p \in [0, \infty)$, i.e. for all t, $0 \leq t \leq 1$,

$$\zeta_{tp+(1-t)p'} \geq t\zeta_p + (1-t)\zeta_{p'} \quad \text{for } p, p' \geq 0$$

Proof: For simplicity we will give the proof for $Re \rightarrow \infty$ first. Note that

$$\begin{aligned}
S_{tp+(1-t)p'}(\mathbf{r}) &= \langle|\delta\mathbf{v}(\mathbf{r})|^{tp+(1-t)p'}\rangle \\
&= \langle|\delta\mathbf{v}(\mathbf{r})|^{tp} \cdot |\delta\mathbf{v}(\mathbf{r})|^{(1-t)p'}\rangle \\
&\leq \langle|\delta\mathbf{v}(\mathbf{r})|^{tp\cdot\frac{1}{t}}\rangle^t \cdot \langle|\delta\mathbf{v}(\mathbf{r})|^{(1-t)p'\cdot\frac{1}{1-t}}\rangle^{1-t} \text{ by Hölder inequality} \\
&= \langle|\delta\mathbf{v}(\mathbf{r})|^{p}\rangle^t \cdot \langle|\delta\mathbf{v}(\mathbf{r})|^{p'}\rangle^{(1-t)} \\
&= [S_p(\mathbf{r})]^t[S_{p'}(\mathbf{r})]^{(1-t)}
\end{aligned}$$

Thus, for $r < L$,

$$\frac{\ln S_{tp+(1-t)p'}(\mathbf{r})}{\ln(r/L)} \geq t \cdot \frac{\ln S_p(\mathbf{r})}{\ln(r/L)} + (1-t) \cdot \frac{\ln S_{p'}(\mathbf{r})}{\ln(r/l)}$$

Taking the limit as $r \rightarrow 0$ of both sides gives the result $\zeta_{tp+(1-t)p'} \geq t\zeta_p + (1-t)\zeta_{p'}$. QED!

One important consequence is the following:

[1] In fact, it is only proved for the Navier-Stokes solution that $\sigma_p = 1$ for $p \leq 2$!

Corollary 1: The exponent $\sigma_p = \zeta_p/p$ is non-increasing in p.

Proof: Note first that

$$\zeta_0 = 0$$

since $S_0(\mathbf{r}) = \langle |\delta \mathbf{v}(\mathbf{r})|^0 \rangle = \langle 1 \rangle = 1!$ Thus, take any $p' \geq p$ and write

$$p = (1 - \tfrac{p}{p'}) \cdot 0 + \tfrac{p}{p'} \cdot p'$$

so that

$$\zeta_p \geq \underbrace{(1 - \frac{p}{p'})\zeta_0}_{=0} + \tfrac{p}{p'}\zeta_{p'}$$

or

$$\sigma_p = \tfrac{\zeta_p}{p} \geq \tfrac{\zeta_{p'}}{p'} = \sigma_{p'} \quad \underline{\text{QED}}$$

On the other hand, we have

Proposition 2: If $\mathbf{v}$ is bounded, then ζ_p is non-decreasing in p.

Proof: if $\mathbf{v} \in L_\infty$, then for all $x \in V$,

$$|\delta \mathbf{v}(\mathbf{r}; \mathbf{x})| \leq 2\|\mathbf{v}\|_\infty$$

Taking any $p' \geq p$,

$$\begin{aligned} \langle |\delta \mathbf{v}(\mathbf{r})|^{p'} \rangle &= \langle |\delta \mathbf{v}(\mathbf{r})|^{p'-p} \cdot |\delta \mathbf{v}(\mathbf{r})|^p \rangle \\ &\leq (2\|\mathbf{v}\|_\infty)^{p'-p} \cdot \langle |\delta \mathbf{v}(\mathbf{r})|^p \rangle \end{aligned}$$

so that

$$\frac{\ln\langle |\delta \mathbf{v}(\mathbf{r})|^{p'} \rangle}{\ln(r/L)} \geq \frac{\ln\langle |\delta \mathbf{v}(\mathbf{r})|^{p} \rangle}{\ln(r/L)} + (p' - p)\frac{\ln(2\|\mathbf{v}\|_\infty)}{\ln(r/L)}$$

The last term vanishes for $r \to 0$, so that taking the $\liminf$ as $r \to 0$ of both sides gives

$$\zeta_{p'} \geq \zeta_p \quad \underline{\text{QED}}.$$

Again, there is a corollory for σ_p:

Corollary 2: If $\mathbf{v}$ is bounded, then $\sigma_p \geq 0$ for all $p > 0$.

Proof: For $p > 0$, $\zeta_p \geq \zeta_0 = 0$. Thus, $\sigma_p = \frac{\zeta_p}{p} \geq 0$. QED

It follows from Corollaries 1 & 2 that

$$\lim_{p\to\infty} \sigma_p = \sigma_\infty \geq 0$$

must exist, if $\mathbf{v}$ is bounded. It is more usual to denote

$$h_{min} = \sigma_\infty,$$

which is called the minimum Hölder exponent of $\mathbf{v}$ in V. It is the most singular behavior of the flow velocity $\mathbf{v}$ that determines h_{min}. For example, for Burgers equation

$$\sigma_p = \begin{cases} 1 & 0 < p < 1 \\ 1/p & p \geq 1 \end{cases}$$

so that

$$h_{min} = \lim_{p\to\infty} \frac{1}{p} = 0$$

corresponding to shock discontinuities in $u(x,t)$. The relation $\sigma_p \to h_{min}$ can also be stated as

$$\zeta_p \sim p h_{min}, \ \ p \to \infty,$$

at least if $h_{min} > 0$.

Note that σ_p and ζ_p can be shown to be nonnegative under an even weaker assumption for each specific $p \geq 1$. That is,

$$\forall p \geq 1, \mathbf{v} \in L_p \implies \sigma_p \geq 0.$$

The proof is essentially the same, since the triangle inequality (only valid for $p \geq 1$!) implies that

$$\|\delta\mathbf{v}(\mathbf{r})\|_p \leq 2\|\mathbf{v}\|_p$$

and thus

$$\sigma_p = \liminf_{r\to 0} \frac{\ln \|\delta\mathbf{v}(\mathbf{r})\|_p}{\ln(r/L)} \geq \liminf_{r\to 0} \frac{\ln(2\|\mathbf{v}\|_p)}{\ln(r/L)} = 0$$

This means that, in order to get a negative σ_p or ζ_p, it must be that $\mathbf{v} \notin L_p$! In that case, even the definition of σ_p and ζ_p that we gave above are invalid, since we always assumed that $\mathbf{v} \in L_p$. It is possible to give a more general definition that allows negative exponents σ_p or ζ_p, for example, by using wavelet coefficients in place of velocity increments. We shall discuss this a little bit later.

If we go back to the proof that ζ_p is non-decreasing for bounded $\mathbf{v}$, we derived

$$\langle|\delta\mathbf{v}(\mathbf{r})|^{p'}\rangle \;\leq\; (2\|\mathbf{v}\|_\infty)^{p'-p}\langle|\delta\mathbf{v}(\mathbf{r})|^{p}\rangle.$$

If we assume scaling, so that $S_p(\mathbf{r}) \cong u_{rms}^p (r/L)^{\zeta_p}$ for $\eta_p \ll r \ll L$, then

$$\begin{aligned} u_{rms}^{p'}(r/L)^{\zeta_{p'}} &\leq (2u_{max})^{p'-p}\cdot u_{rms}^p(r/L)^{\zeta_p} \\ \Longrightarrow (r/L)^{\zeta_{p'}-\zeta_p} &\leq (\frac{2u_{max}}{u_{rms}})^{p'-p} \end{aligned}$$

If we take $\eta = \max \eta_p, \eta_{p'} \to 0$ as $Re \to \infty$, then by taking r/L sufficiently small as $Re \to \infty$ we get that

$$\lim_{Re\to\infty} \frac{u_{max}}{u_{rms}} = +\infty$$

whenever $\zeta_{p'} < \zeta_p$ for some $p' > p$. Even if we assume that $u_{rms} < +\infty$ as $Re \to \infty$, we see that $u_{max} \to \infty$. This is the contrapositive of the statement that $\mathbf{v}$ bounded implies ζ_p increasing: if $\zeta_{p'} < \zeta_p$ for even one value $p' > p$, then $u_{max} \to \infty$ as $Re \to \infty$. Uriel Frisch, Section 8.4 argues that incompressible Navier-Stokes equation will not be a physically valid model at very large Reynolds number if it leads to

$$\zeta_{p'} < \zeta_p, \text{ for some } p' > p.$$

His reasoning is that

$$u_{max} \to \infty \text{ as } Re \to \infty$$

in that case, so that the maximum Mach number

$$Ma_{max} = \tfrac{u_{max}}{c} \to \infty,$$

violating the assumption of small Mach number in the derivation of the incompressible Navier-Stokes from either molecular dynamics or from compressible flow.

However, the work of Quastel & Yan (1998) cited earlier shows that this argument is incorrect. They derived the incompressible Navier-Stokes equation rigorously from a stochastic lattice gas in the limit where a bulk Mach number and Knudsen number are small. Their derivation is valid even if the Navier-Stokes equation leads to singular Leray solutions which, it is known, must have $u_{max} = +\infty$. Thus, it is proved that incompressible Navier-Stokes (INS) may be physically valid even if $u_{max} = +\infty$. This is related to partial regularity results for the

Leray solutions of the Navier-Stokes equation, which imply that the fractal (actually, parabolic Hausdorff) dimension of the singular set is small, much less than the dimension of spacetime. In particular, the set where $u_{max} = +\infty$ is proved to have zero measure (and possibly is empty.) Thus, the assumptions under which INS are derived — in a distribution sense! — are not violated by singular Leray solutions.

It is worth emphasizing, however, that the proof of Quastel & Yan (1998) does not, in fact, justify the validity of deterministic Navier-Stokes equations for turbulent flows, in practice. Their argument is based on the scaling symmetry of the deterministic equations discussed in Chapter II:

$$\mathbf{v} \to \mathbf{v}' = \lambda \mathbf{v}, \quad \mathbf{x} \to \mathbf{x}' = \lambda^{-1}\mathbf{x}, \quad t \to t' = \lambda^{-2} t$$

which maps an incompressible Navier-Stokes solution $\mathbf{v}(\mathbf{x}, t)$ to another solution $\mathbf{v}'(\mathbf{x}', t')$ with the same Reynolds number $Re' = Re$ and with molecular viscosity ν also unchanged. This self-similarity is broken, however, by thermal noise. For the case of turbulent flows where Taylor's relation holds:

$$\begin{aligned} \varepsilon &\sim U^3/L \to \varepsilon' = \lambda^4 \varepsilon \\ u_\eta &= (\nu\varepsilon)^{1/4} \to u'_\eta = \lambda u_\eta, \\ \eta &= \nu^{3/4}\varepsilon^{-1/4} \to \eta' = \lambda^{-1}\eta \\ \theta_\eta &= k_B T/\rho u_\eta^2 \eta^3 \to \theta'_\eta = \lambda \theta_\eta \end{aligned}$$

Because θ_η decreases with λ, it is possible, in principle, to observe deterministic Navier-Stokes predictions in the dissipation range of turbulent flows by taking $\mathbf{v} \to \mathbf{v}' = \lambda \mathbf{v}$, $\mathbf{x} \to \mathbf{x}' = \lambda^{-1}\mathbf{x}$ with $\lambda \ll 1$. However, in practice, λ must be chosen unrealistically small. If relation $x^2 e^x = 1/\theta_\eta$ is solved for $x = k_c \eta$, then satisfying it for $x' = 2x$, requires $1/\lambda = \theta_\eta/\theta'_\eta = 4e^x$. For example, in the case of the ABL we argued in Chapter II that thermal noise would be relevant already at a length-scale $\ell \sim \eta/11$. To make the deterministic predictions valid down to $\ell \sim \eta/22$ would require that the integral length be made $4e^{11} = 240,000$ times larger and r.m.s. velocities 240,000 times weaker!

Lastly, let us comment on the special role of K41 among all possible scaling laws

$$S_p(\mathbf{r}) \sim C_p u_{rms}^p (\tfrac{r}{L})^{\zeta_p}.$$

Using $u_{rms} \cong (\langle\varepsilon\rangle L)^{1/3}$ and setting $\zeta_p = p/3 + \delta\zeta_p$, gives

$$S_p(\mathbf{r}) \sim C_p(\langle\varepsilon\rangle r)^{p/3}(\tfrac{r}{L})^{\delta\zeta_p}. \qquad (\star)$$

Thus we see that K41 with $\zeta_p = p/3$ or $\delta\zeta_p = 0$ for all $p \geq 0$ is unique in that it is the only possible scaling in which $S_p(\mathbf{r})$ is independent of L, depending only upon $\langle\varepsilon\rangle$. K41 predictions are often "derived" by dimensional considerations, assuming that $\langle\varepsilon\rangle$ is the only relevant parameter.

More generally, $S_p(\mathbf{r})$ should be expected to depend upon both $\langle\varepsilon\rangle$ and L, even for $r \ll L$. This means that the small-scales "remember" not only the energy flux from the large-scales but also

$$N = \log_2(\tfrac{L}{r})$$

the number of "cascade steps" (by factors of 2) in going from the length L to the length r. Intuitively, intermittency corresponds to fluctuations that build up and become larger and larger as r decreases further below L.

In physics, scaling laws like $(\star)$ above are called anomalous because the dimensional scaling based upon $\langle\varepsilon\rangle$ breaks down. K41 scaling implicitly assumes that the limit $L/r \to \infty$ exists, which need not be true (e.g. Burgers equation!) The exponent $\delta\zeta_p$ that describes this divergence is called an anomalous dimension. Unfortunately, $\delta\zeta_p$ cannot be obtained by a dimensional analysis. The renormalization group is a general tool developed in physics to calculate such anomalous dimensions. See N. Goldenfeld, *Lectures on Phase Transitions and the Renormalization Group* (Westview Press, 1992).