Problem 1. (a) Write

\[ p = \frac{p^2}{p} = \frac{q^2 + (p^2 - q^2)}{p} = \left( \frac{q}{p} \right) q + \left( \frac{p - q}{p} \right) (p + q) \]

Then, by concavity,

\[ 5p \geq \left( \frac{q}{p} \right) 5q + \left( \frac{p - q}{p} \right) 5p + q \]

\[ \Rightarrow \quad p \geq q + (p - q) \frac{5p + q}{p - q} \]

\[ \Rightarrow \quad q \geq q + (p - q) \left( \frac{5p + q - 5p}{p - q} \right) \]

\[ \Rightarrow \quad q \geq \frac{5p - 5q}{p - q} \geq \frac{5p + q - 5p}{q} \]

(b) Taking the limit \( q \to 0 \) gives

\[ \sigma_p = \frac{5p}{p} \geq \frac{d \sqrt{p}}{dp} = \frac{1}{2} \frac{5p - 5q}{p - q} \]

Taking the limit \( q \to p \) gives

\[ h_p = \frac{d \sqrt{p}}{dp} \geq \frac{5p - 5q}{p - q} = 2 \left( \frac{5p}{2p} \right) - \frac{5p}{p} \]

\[ = 2 \sigma_{2p} - \sigma_p \]
(c) It was shown in the classnotes that

$$\lim_{p \to \infty} \sigma_p = h_{\min}.$$ 

Taking the limit as $p \to \infty$ of

$$2\sigma_p - \sigma_p \leq h_p \leq \sigma_p$$

gives

$$h_{\min} = 2h_{\min} - h_{\min} \leq \lim_{p \to \infty} h_p \leq h_{\min}$$

$$\implies \lim_{p \to \infty} h_p = h_{\min}.$$ 

Problem 2. (a) For $K62$,

$$\varepsilon_p = \frac{p}{3} - \frac{M}{18}(p^2 - 3p)$$

$$\implies h_p = \frac{1}{3} - \frac{M}{18}(2p - 3) \quad \text{or} \quad \frac{1}{3} \frac{1}{3} \frac{1}{3} \frac{1}{3} - \frac{1}{3} \frac{1}{3} \frac{1}{3} \frac{1}{3}$$

if $\mu = 1/4$

For $SL$,

$$\varphi = \frac{p}{q} + 2 \left[ 1 - \left( \frac{2}{3} \right)^{P/3} \right] = \frac{p}{q} + 2 - 2\exp \left[ \frac{1}{3} \ln \left( \frac{2}{3} \right) \cdot p \right]$$

$$\implies h_p = \frac{1}{q} - \frac{2}{3} \ln \left( \frac{2}{3} \right) \cdot \left( \frac{2}{3} \right)^{P/3}$$

$$h_p = \frac{1}{q} - \left( \frac{2}{3} \right)^{P/3 + 1} \ln \left( \frac{2}{3} \right)$$
For MF

\[ S_p = \frac{a_p}{v + c_p} \]

\( \Rightarrow \)

\[ h_p = \frac{(b + c_p)a - (c_p)c}{(b + c_p)^2} = \frac{ab}{(b + c_p)^2} \]

(b) See the separate figure for the plots of \( S_p \) and \( h_p \) in the three models. The predictions for \( S_p \) virtually collapse in the range \( 0 \leq p \leq 5 \), to the eye, and only separate for \( p > 5 \). By contrast, the predictions for \( h_p \) are distinct over the whole range and only coincide (nearly) for \( p \approx 1, 2 \). Thus, the model predictions for \( h_p \) are more distinctive than those for \( S_p \). Of course, it is more difficult to get accurate results for \( h_p \) from experiments & simulations than it is for \( S_p \), because of less of significance in differentiating empirical data.

(c) For K62

\[ h_{\text{min}} = \lim_{p \to 0} \left[ \frac{1}{3} - \frac{h}{18} (2p - 3) \right] = -\infty \]

for SL

\[ h_{\text{min}} = \lim_{p \to \infty} \left[ \frac{1}{9} - \left( \frac{2}{3} \right)^{\frac{p}{2} + 1} \ln \left( \frac{2}{3} \right) \right] = \frac{1}{9} \]

and for MF

\[ h_{\text{min}} = \lim_{p \to \infty} \left[ \frac{ab}{(b + c_p)^2} \right] = 0 \]
Theoretical Predictions for $\zeta_p$

- K41
- K62
- SL
- MF
Theoretical Predictions for $h_p = \frac{d\xi_p}{dp}$

- K41
- K62
- SL
- MF
Problem 3. A Matlab script `scalexp.m` is printed on the following page which generates all of the numerical output for this problem.

(a) See the plots on the following pages. A table of values of exponents \(5^p, 5^N\) and amplitudes \(C_p, C_N\) for \(p = 1/2, \ldots, 10\) is also on the following pages. As one can see from the plots of the structure functions, they start to show rather large "wiggles" or oscillations for \(p = 3, 8\) and higher. These are presumably due to statistical errors arising from insufficient number of samples to get converged results for high-order moments. Only the values that we have calculated for \(p = 1/2, \ldots, 6\) have reasonable reliability.

(b) One can see from the table of values and the plot of \(5^L, 5^N\) versus \(p\) that

\[5^N < 5^L\] for \(p \geq 3\)

so that the transverse velocity differences are more intermittent.

(c) In the plot of \(5^p\) versus \(p\) and comparison with the predictions of the four theories, there is a rather clear disagreement of the K41 prediction \(p/3\) with the empirical exponents. On the other hand, the other three theories are in reasonable agreement with the data for \(p = 1/2, \ldots, 6\) and difficult to distinguish from one another.
<table>
<thead>
<tr>
<th>$p$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_p^L$</td>
<td>0.3735</td>
<td>0.7153</td>
<td>1.0240</td>
<td>1.2953</td>
<td>1.5254</td>
<td>1.7137</td>
<td>1.8640</td>
<td>1.9831</td>
<td>2.0797</td>
<td>2.1622</td>
</tr>
<tr>
<td>$C_p^N$</td>
<td>0.3833</td>
<td>0.7147</td>
<td>0.9954</td>
<td>1.2266</td>
<td>1.4113</td>
<td>1.5551</td>
<td>1.6658</td>
<td>1.7518</td>
<td>1.8211</td>
<td>1.8802</td>
</tr>
<tr>
<td>$C_p^L$</td>
<td>0.8617</td>
<td>1.1669</td>
<td>1.9960</td>
<td>3.9405</td>
<td>8.5223</td>
<td>19.5643</td>
<td>46.8746</td>
<td>116.4367</td>
<td>299.7270</td>
<td>801.4197</td>
</tr>
</tbody>
</table>
absolute structure function scaling exponents from JHU database

- longitudinal
- transverse
- K41
absolute longitudinal structure function scaling exponents: JHU data vs. theory

- JHU data
- K41
- K62
- SL
- VY
absolute longitudinal structure function, order $p=1$
absolute longitudinal structure function, order p=2
absolute longitudinal structure function, order p=3
absoute longitudinal structure function, order p=4
absolute longitudinal structure function, order $p=6$
absolute longitudinal structure function, order p=7
absolute longitudinal structure function, order p=9
absolute longitudinal structure function, order $p=10$
absolute transverse structure function, order p=1
absolute transverse structure function, order p=2
absolute transverse structure function, order p=3
absolute transverse structure function, order p=4
absolute transverse structure function, order $p=5$
absolute transverse structure function, order $p=6$
absolute transverse structure function, order p=7

separation r/L
absolute transverse structure function, order $p=8$
absolute transverse structure function, order p=9
absolute transverse structure function, order p=10
clear all
urms=0.606;
LL=1.364;
etat=0.00280;
FROM=1;
T0=192;
NUM=T0-FROM+1;
ASFL=zeros(512,10); ASFN=zeros(512,10);
for k=FROM:T0
    vel=load(sprintf('DATA/RUN%d/velocity.dat',k));
    for ii=1:512
        dv=circshift(vel,1-ii)-vel;
dVN1=dv(:,2)/urms;
dVN2=dv(:,3)/urms;
dvL=dv(:,4)/urms;
        for pp=1:10
            ASFL(ii,pp)= ASFL(ii,pp)+sum(abs(dvL).^pp)/1024;
            ASFN(ii,pp)= ASFN(ii,pp)+sum(abs(dvN1).^pp+abs(dvN2).^pp)/1024;
        end
    end
end
ASFL=ASFL/NUM; ASFN=ASFN/NUM/2;
rr=(0:511)'; rr=rr*2*pi/1024; rr=rr/LL;
save ASFL
save ASFN
for pp=1:10
    PPL=polyfit(log(rr(18:63)),log(ASFL(18:63,pp)),1);
zL(pp)=PPL(1); CL(pp)=exp(PPL(2));
PNN=polyfit(log(rr(18:163)),log(ASFN(18:163,pp)),1);
zN(pp)=PNN(1); CN(pp)=exp(PNN(2));
figure; loglog(rr,ASFL(:,pp),'-b',rr,CL(pp)*rr.^zL(pp),'-r')
xlabel('separation r/L')
title(sprintf('absolute longitudinal structure function, order p=%d',pp))
fgnm=['figs/LongStructFun-p',num2str(pp),'.pdf'];
print(gcf,'-dpdf',fgnm)
figure; loglog(rr,ASFN(:,pp),'-b',rr,CN(pp)*rr.^zN(pp),'-r')
xlabel('separation r/L')
title(sprintf('absolute transverse structure function, order p=%d',pp))
fgnm=['figs/TransvStructFun-p',num2str(pp),'.pdf'];
print(gcf,'-dpdf',fgnm)
end

save zL
save CL
save zN
save CN

zL=[0, zL]; zN=[0, zN];
ppp=0:10;
figure; plot(ppp,zL, '--b', ppp,zN, '--g', ppp, ppp/3, '-k');
xlabel('order p')
ylabel('scaling exponent \( \zeta_p \)')
title('absolute structure function scaling exponents from JHU database')
legend('longitudinal', 'transverse', 'K41', 'Location', 'NorthWest')
axis([0 6 0 2])
fignm=[]'figs/StrucFunExps.pdf'
print(gcf, '-dpdf', fgnm)

figure; plot(ppp,zL, '*'k', ppp, ppp/3, '-k', ppp, ppp/3-(ppp.^2-3*ppp)/72, '-b', ...
ppp, ppp/9+2*(1-(2/3).^((ppp/3)), '--g', ppp, 0.185*ppp./((0.473+.0275*ppp), '-r')
xlabel('order p')
ylabel('scaling exponent \( \zeta_p \)')
title('absolute longitudinal structure function scaling exponents: JHU data vs. theory')
legend('JHU data', 'K41', 'K62', 'SL', 'VY', 'Location', 'NorthWest')
axis([0 6 0 2])
fignm=[]'figs/Numereics-vs-Theory.pdf'
print(gcf, '-dpdf', fgnm)
Problem 4. Starting with
\[ \partial_t u + \partial_x \left( \frac{1}{2} u^2 \right) = \nu \partial_x^2 u \]
\[ \partial_t u^1 + \partial_x \left( \frac{1}{2} u^1 \right) = \nu \partial_x^2 u^1 \]
we get
\[ \partial_t (uu^1) + u^1 \partial_x \left( \frac{1}{2} u^2 \right) + u \partial_x \left( \frac{1}{2} u^2 \right) = \nu u^1 \partial_x^2 u + \nu u \partial_x^2 u^1 \]

Note first that
\[ u^1 \partial_x^2 u + u \partial_x^2 u^1 = \partial_x \left[ u^1 \partial_x u^1 + u \partial_x u^1 \right] - 2 \partial_x u \partial_x u^1 \]
\[ = \partial_x^2 (uu^1) - 2 \partial_x u \partial_x u^1 \]

For the cubic term,
\[ u \partial_x (u^1 u^2) + u^1 \partial_x (u^2) \]
\[ = u \partial_x (u^1 u^2) + \partial_x (u^1 u^2) - u^2 \partial_x u^1 \]
\[ = 2uu^1 \partial_x u^1 - u^2 \partial_x u^1 + \partial_x (u^1 u^2) \]
\[ = \left[ u^1 u^2 - (u^1 - u)^2 \right] \partial_x u^1 + \partial_x (u^1 u^2) \]
\[ = -(6u)^2 \partial_x u^1 + u^1 u^2 \partial_x u^1 + \partial_x (u^1 u^2) \]
\[ = -(6u)^2 \partial_x u + \partial_x \left( \frac{1}{3} u^3 \right) + \partial_x (u^1 u^2) \]
\[ = -\frac{1}{3} \partial_x (6u)^3 + \partial_x \left( u^1 u^2 + \frac{1}{3} u^3 \right) \]

Combining these results then gives

(carr1d)
\[ \frac{\partial}{\partial t} (uu') - \frac{1}{6} \partial_t (\delta u)^3 + \partial_x \left( \frac{1}{2} u'u^2 + \frac{1}{6} u'^3 \right) \]

\[ = \nu \partial_x^2 (uu') - 2 \nu \partial_x u \partial_x u' \]

\[ \text{or} \]

\[ \frac{\partial}{\partial t} \left( \frac{1}{2} uu' \right) + \partial_x \left( \frac{1}{4} u'u^2 + \frac{1}{12} u'^3 - \nu \partial_x \left( \frac{1}{2} uu' \right) \right) \]

\[ = \frac{1}{12} \partial_t (\delta u)^3 - \nu \partial_x u \partial_x u' \]

\[ \boxed{ \text{QED} } \]