1. This problem studies the decreasing function $h_p = d\zeta_p/dp$. As we shall see later in studying the multifractal formalism, $h_p$ has an interpretation as the H"older singularity of the velocity field which makes the dominant contribution to the inertial-range scaling of the structure function of order $p$.

(a) Show that the concave function $\zeta_p$ satisfies the inequality

$$\frac{\zeta_{p+q} - \zeta_p}{q} \leq \frac{\zeta_p - \zeta_q}{p - q},$$

for $0 \leq q \leq p$. *Hint*: $p$ lies between $q$ and $p + q$.

(b) Use part (a) to show that

$$2\sigma_{2p} - \sigma_p \leq h_p \leq \sigma_p,$$

where $\sigma_p = \zeta_p/p$.

(c) Conclude that $\lim_{p \to \infty} h_p = h_{\text{min}}$.

2. In this problem we shall consider several theoretical predictions for the scaling exponents $\zeta_p$ of the longitudinal velocity structure functions. These are the predictions by the lognormal model of A. N. Kolmogorov, J. Fluid Mech. 13 82 (1962)

$$(K62) \quad \zeta_p = \frac{p}{3} - \frac{\mu}{18}p(p - 3)$$

with $\mu = 0.25$, the log-Poisson model of Z.-S. She and E. Lévêque, Phys. Rev. Lett. 72 336 (1994)

$$(SL) \quad \zeta_p = \frac{p}{9} + 2 \left[ 1 - \left( \frac{2}{3} \right)^{p/3} \right]$$

and the “mean-field theory” of V. Yakhot, Phys. Rev. E 63 026307 (2001)

$$(MF) \quad \zeta_p = \frac{ap}{b + cp}$$

with $a = 0.185, b = 0.473$ and $c = 0.0275$.

(a) For each of these theoretical models for $\zeta_p$ calculate the prediction for $h_p$.

(b) Plot the three models for $\zeta_p$ over the range $0 \leq p \leq 10$ and then do likewise for the three predictions of $h_p$. Which of these better distinguishes the different models?

(c) Calculate the value $h_{\text{min}}$ predicted by each of the three models.
3. This problem studies the scaling exponents of the velocity structure-functions empirically, using data from a 1024^3 simulation in a periodic box of forced, steady-state turbulence stored in an online database at the Johns Hopkins University:

http://turbulence.pha.jhu.edu/docs/README-isotropic.pdf

The velocity fields in 192 vertical “cuts” through this database have been downloaded and made available at the course website in a zipped directory:

http://www.ams.jhu.edu/~eyink/Turbulence/TURBDATA.zip

When uncompressed, the directory “DATA” contains folders “RUN1”, ..., “RUN192” storing velocity-field data harvested along the cuts. These can be easily read in Matlab, for example, with the command

\[
\text{vel}=\text{load}(\text{sprintf}('\text{DATA/RUN%d/velocity.dat}',k));
\]

for \(k=1:192\) with the resulting array \(\text{vel}\) containing the z-position in the first column, and the x-, y- and z-components of the velocity field in the second, third, and fourth columns, respectively. The array created by the Matlab command

\[
\text{dv}=\text{circshift}(\text{vel},1-\text{ii})-\text{vel};
\]

for \(\text{ii}=1:512\) then creates the velocity-differences with increments \(\text{rr}=\text{ii}*2*\pi/1024\) directed along the z-axis.

(a) Using this data, calculate numerically the absolute velocity-structure functions

\[
S_p^X(r) = \langle |\delta v^X(r)|^p \rangle,
\]

for \(p = 1, 2, ..., 10\) and \(X = L\) (longitudinal), \(C = N\) (transverse) and plot them log-log. By a least-squares fit in log-log over the two-octave range \(r = (0.0561)L\) to \(r = (0.2045)L\), estimate the constants \(C_p^X\) and scaling exponents \(\zeta_p^X\) in the inertial-range scaling laws

\[
S_p^X(r) \sim C_p^X u_{\text{rms}}^p (r/L)^{\zeta_p^X}.
\]

Based upon your plots, what is the largest value of \(p\) for which you believe that your results for exponents and amplitudes are reliable?

(b) Plot your results for \(\zeta_p^L\) and \(\zeta_p^N\) versus \(p\), for \(p = 0, 1, ..., 6\). Which velocity-differences are more intermittent, the longitudinal or the transverse?

(c) Repeat the plot of longitudinal exponents \(\zeta_p^L\) versus \(p\), for \(p = 0, 1, ..., 6\) and plot also the predictions of K41 (\(\zeta_p^L = p/3\)) and of the three models in Problem #2 (log-normal, log-Poisson, mean-field). Are the predictions distinguishable by the data?
4. For one-dimensional decaying Burgers equation

$$\partial_t u + u \partial_x u = \nu \partial_x^2 u,$$

derive the following balance relation for the “point-split” energy density

$$\partial_t \left( \frac{1}{2} uu' \right) + \partial_x \left[ \frac{1}{4} u'u^2 + \frac{1}{12} u^3 - \nu \partial_x \left( \frac{1}{2} uu' \right) \right] = \partial_r \left( \frac{1}{12} \delta u^3 \right) - \nu (\partial_x u)(\partial_x u'),$$

where $u = u(x, t), u' = u(x + r, t)$ and $\delta u = u' - u = u(x + r, t) - u(x, t)$. 