Homework No.7, 553.793, April 15, 2022.

1. (a) If the resolved entropy density is defined as $s_\ell = \rho c_p \ln T_\ell$ for the incompressible Navier-Stokes-Fourier system, then show that in the limit $\nu, \kappa \to 0$ it satisfies the balance equation

$$\partial_t s_\ell + \nabla \cdot [s_\ell \nabla + \beta_\ell \tau_\ell(u, v)] = \nabla \beta_\ell \cdot \tau_\ell(u, v) + \rho \beta_\ell \varepsilon_\ell,$$

where $u = \rho c_p \frac{T_\ell}{T}$ is the internal energy per volume of the fluid, $\beta_\ell = 1/T_\ell$, and $\varepsilon_\ell$ is the coarse-grained energy dissipation rate (per mass), which is assumed not to vanish as $\nu, \kappa \to 0$. You may find the results of Homework #4, Problem 7(b) helpful to simplify the coarse-grained equations in the limit $\nu, \kappa \to 0$.

(b) A more useful definition of resolved entropy density is

$$s_\ell^* = s_\ell + \beta_\ell \rho k_\ell,$$

where $k_\ell$ is the unresolved kinetic energy density (per mass), which satisfies

$$\partial_t \rho k_\ell + \nabla \cdot \left[ \rho k_\ell \nabla + \tau_\ell(P, v) + \frac{1}{2} \rho \tau_\ell(v_i, v_i, v) \right] = \rho \Pi_\ell - \rho \varepsilon_\ell$$

in the limit $\nu \to 0$. Using the latter, derive the balance equation

$$\partial_t s_\ell^* + \nabla \cdot [s_\ell^* \nabla + \beta_\ell \rho k_\ell] = \rho \beta_\ell \Pi_\ell + (\nabla \beta_\ell) \rho k_\ell + \nabla \beta_\ell \cdot \mathbf{q}_\ell$$

where we have defined the “turbulent heat-transport vector” as

$$\mathbf{q}_\ell := \tau_\ell(h, v) + \frac{1}{2} \rho \tau_\ell(v_i, v_i, v)$$

with $h = u + P = \rho (c_p T + p)$ the thermodynamic enthalpy per volume.

2. Assume that velocity field $v$ is Hölder continuous with an exponent $0 < h < 1$, which is the condition under which energy cascade is scale-local. Under the same condition, use the expression

$$\Lambda_\ell(v, v, v) = -2(\nabla \times \nabla \ell) \cdot f_\ell^\delta(v, v)$$

for helicity flux to decide whether helicity cascade is also scale-local. **Hint:** Try to derive upper bounds on $\Lambda_\ell(v, v, \nabla \Delta)$ for $\Delta > \ell$ and $\Lambda_\ell(v, v, v_\delta')$ for $\delta < \ell$, etc.
3. Consider the Khokhlov sawtooth solution of Burgers equation

\[ u^\nu(x,t) = \frac{1}{t} \left[ x - L \tanh \left( \frac{Lx}{2\nu t} \right) \right]. \]

on the real line.

(a) Show that this converges in the inviscid limit \( \nu \to 0 \) pointwise to

\[ u(x,t) = \left( \frac{x + L}{t} \right) \theta(-x) + \left( \frac{x - L}{t} \right) \theta(x), \]

where

\[ \theta(x) = \begin{cases} 
0 & x < 0 \\
\frac{1}{2} & x = 0 \\
1 & x > 0 
\end{cases} \]

is the Heaviside step function.

(b) Use the well-known facts that \( \theta(-x) + \theta(x) = 1 \) and \( \theta'(x) = \delta(x) \) to show that

\[ \partial_t u + \partial_x \left( \frac{1}{2} u^2 \right) = 0. \]

Thus, \( u \) is a solution of inviscid Burgers equation in the sense of distributions. \( \text{Hint:} \) You should see a cancellation at \( x = 0 \).

(c) Prove the following more formal statement of the result in (b): For any \( C^\infty \) test function \( \varphi(x,t) \) with compact support

\[ \int_{t_0}^\infty dt \int dx \left[ u(x,t) \partial_t \varphi(x,t) + \frac{1}{2} u^2(x,t) \partial_x \varphi(x,t) \right] + \int dx \varphi(x,t_0) u(x,t_0) = 0 \]

with \( t_0 > 0 \). \( \text{Hint:} \) Since \( u(x,t) \) is smooth in the variable \( t \), integrate by parts in time and substitute the expressions for \( u \) and \( \partial_t u \). You should obtain the same cancellation at \( x = 0 \) as in part (b).

4. This problem studies local energy balance for the Khokhlov sawtooth solution \( u \) of inviscid Burgers equation.

(a) Use the same method of argument as in Problem 3 (b) to show that

\[ \partial_t \left( \frac{1}{2} u^2 \right) + \partial_x \left( \frac{1}{3} u^3 \right) = -\frac{1}{12} (\Delta u)^3 \delta(x) \]

in the sense of distributions, where \( \Delta u = 2L/t \) is the velocity discontinuity at \( x = 0 \).

(b) Use the formula for viscous dissipation in the sawtooth

\[ \varepsilon^\nu(x,t) \simeq \frac{L^4}{4\nu t^4} \text{sech}^4 \left( \frac{Lx}{2\nu t} \right), \]

where \( \text{sech} \) is the hyperbolic secant function.
which is asymptotically valid for small \( \nu \), to show that
\[
\lim_{\nu \to 0} \int dx \varphi(x) \varepsilon^\nu(x, t) = \frac{1}{12} (\Delta u)^3 \varphi(0)
\]
for any \( C^\infty \) function \( \varphi \) with compact support. Thus,
\[
\lim_{\nu \to 0} \varepsilon^\nu(x, t) = \frac{1}{12} (\Delta u)^3 \delta(x)
\]
in the sense of distributions, at every time \( t \).

5. This problem discusses the notion of intermediate asymptotics for the example of the Khokhlov sawtooth. Although the viscous solution \( u^\nu \) is smooth everywhere, it “appears” to have a discontinuous jump \( \Delta u \) in velocity at \( x = 0 \) for a long range of length-scales. More precisely, consider the symmetric difference
\[
\delta_S u^\nu(r; t) = u^\nu\left(\frac{r}{2}, t\right) - u^\nu\left(-\frac{r}{2}, t\right),
\]
at \( x = 0 \). Show that when \( L(\Delta u)/\nu \gg 1 \), then an intermediate asymptotics holds for \( \nu/\Delta u \ll |r| \ll L \)
\[
\delta_S u^\nu(r; t) \simeq -(\Delta u) \text{sign}(r) = O(r^0),
\]
although for \( |r| \ll \nu/\Delta u \)
\[
\delta_S u^\nu(r; t) \simeq \frac{r}{t} \left(1 - \frac{L^2}{2\nu t}\right) = O(r^1).
\]