Homework #6

Problem 1. (a) By definition

$$T(u_i,u_j,u_k) = \overline{u_i u_j u_k} - \overline{u_i}\, T(u_j,u_k) - \overline{u_j}\, T(u_i,u_k) - \overline{u_k}\, T(u_i,u_j) - \overline{u_i}\,\overline{u_j}\,\overline{u_k}$$

$$= \overline{u_i u_j u_k} - \overline{u_i}\left[\overline{u_j u_k} - \overline{u_j}\,\overline{u_k}\right] - \overline{u_j}\left[\overline{u_i u_k} - \overline{u_i}\,\overline{u_k}\right] - \overline{u_k}\left[\overline{u_i u_j} - \overline{u_i}\,\overline{u_j}\right] - \overline{u_i}\,\overline{u_j}\,\overline{u_k}$$

$$= \overline{u_i u_j u_k} - \overline{u_i}\,\overline{u_j u_k} - \overline{u_j}\,\overline{u_i u_k} - \overline{u_k}\,\overline{u_i u_j} + 2\overline{u_i}\,\overline{u_j}\,\overline{u_k}.$$

(b) Recall that $W_\ell(\alpha) = \ln Z_\ell(\alpha)$ with

$$Z_\ell(\alpha) = \overline{\left(\exp(\alpha\cdot u)\right)_\ell}.$$

Thus,

$$\overline{u_i}(\alpha) \equiv \frac{\partial}{\partial \alpha_i} W(\alpha) = \frac{1}{Z(\alpha)}\frac{\partial Z}{\partial \alpha_i}(\alpha) = \frac{1}{Z(\alpha)}\overline{\left(u_i e^{\alpha\cdot u}\right)_\ell}$$

Similarly,

$$T_{ij}(\alpha) = \frac{\partial^2}{\partial\alpha_i\partial\alpha_j} W(\alpha)$$

$$= \frac{\partial}{\partial\alpha_j}\left[\frac{1}{Z(\alpha)}\overline{\left(u_i e^{\alpha\cdot u}\right)_\ell}\right]$$

$$= \frac{1}{Z(\alpha)}\overline{\left(u_i u_j e^{\alpha\cdot u}\right)_\ell} - \frac{1}{Z^2(\alpha)}\overline{\left(u_i e^{\alpha\cdot u}\right)_\ell}\,\overline{\left(u_j e^{\alpha\cdot u}\right)_\ell}$$

and

$$T_{ijk}(\alpha) = \frac{\partial^3}{\partial\alpha_i \partial\alpha_j \partial\alpha_k} W(\alpha)$$

$$= \frac{\partial}{\partial\alpha_k}\left[\frac{1}{Z(\alpha)} \overline{(u_i u_j e^{\alpha\cdot u})}_\ell - \frac{1}{Z^2(\alpha)} \overline{(u_i e^{\alpha\cdot u})}_\ell \, \overline{(u_j e^{\alpha\cdot u})}_\ell \right]$$

$$= \frac{1}{Z(\alpha)} \overline{(u_i u_j u_k e^{\alpha\cdot u})}_\ell - \frac{1}{Z^2(\alpha)} \overline{(u_k e^{\alpha\cdot u})}_\ell \, \overline{(u_i u_j e^{\alpha\cdot u})}_\ell$$

$$- \frac{1}{Z^2(\alpha)} \overline{(u_i u_k e^{\alpha\cdot u})}_\ell \, \overline{(u_j e^{\alpha\cdot u})}_\ell - \frac{1}{Z^2(\alpha)} \overline{(u_i e^{\alpha\cdot u})}_\ell \, \overline{(u_j u_k e^{\alpha\cdot u})}_\ell$$

$$+ \frac{2}{Z^3(\alpha)} \overline{(u_i e^{\alpha\cdot u})}_\ell \, \overline{(u_j e^{\alpha\cdot u})}_\ell \, \overline{(u_k e^{\alpha\cdot u})}_\ell .$$

Finally, setting $\alpha = 0$, we get again that (since $Z(0) = 1$)

$$T(u_i, u_j, u_k) = T_{ijk}(\alpha)\big|_{\alpha=0}$$

$$= \overline{u_i u_j u_k} - \overline{u_k}\,\overline{u_i u_j} - \overline{u_i u_k}\,\overline{u_j} - \overline{u_j u_k}\,\overline{u_i} + 2\overline{u_i}\,\overline{u_j}\,\overline{u_k} .$$

(c) In Homework #4, Problem 2(b) it was shown that 3rd-order cumulants are always shift-invariant. Since

$$T_\ell(u_i, u_j, u_k) = \langle (\sigma u_i)(\sigma u_j)(\sigma u_k) \rangle^c_\ell$$

Employing the shift $\sigma u_i(r;x) \to \sigma u_i(r;x) - u_i(x) = \delta u_i(r;x)$,

$$T_\ell(u_i, u_j, u_k) = \langle (\delta u_i)(\delta u_j)(\delta u_k) \rangle^c_\ell$$

$$= \langle \delta u_i \delta u_j \delta u_k \rangle_\ell - \langle \delta u_i \rangle_\ell \langle \delta u_j \delta u_k \rangle_\ell - \langle \delta u_j \rangle \langle \delta u_i \delta u_k \rangle_\ell$$

$$- \langle \delta u_k \rangle_\ell \langle \delta u_i \delta u_j \rangle_\ell + 2 \langle \delta u_i \rangle_\ell \langle \delta u_j \rangle_\ell \langle \delta u_k \rangle_\ell .$$

Problem 2. (a) Using the relation

$$\int_0^\infty dk\, E_{f_*}(k) = \langle |f_*|^2 \rangle$$

we see that $\langle |f_*|^2 \rangle < +\infty$ if $E_{f_*}(k) \sim k^{-n}$ for $n>1$, but that $\langle |f_*|^2 \rangle = +\infty$ if $E_{f_*}(k) \sim k^{-n}$ for $n \leq 1$. Thus, f is inertial-range if $n>1$ and dissipation-range if $n \leq 1$.

(b) Since

$$\langle |u_*|^2 \rangle < +\infty$$

(certainly in decaying turbulence), u_ν is inertial-range. However,

$$\langle |\nabla u_\nu|^2 \rangle = \frac{\varepsilon}{\nu} \longrightarrow \infty$$

as $\nu \to 0$, so that (∇u_ν) is dissipation-range. Since the pressure has

$$\| \delta p_*(r) \|_q \leq (\text{const.}) \| \delta u_*(r) \|_{2q}^2 = O(r^{\zeta_{2q}^{(u)}/q})$$

it follows that p_* is even more regular than u_*. Thus, pressure p_ν is inertial-range. However, the evidence is that $\zeta_{2q}^{(u)} \leq \frac{2}{3} q$, $q \geq 3/2$ (to account for energy dissipation) so that

$$\sigma_q^{(p)} = \frac{\zeta_q^{(p)}}{q} \cong \sigma_{2q}^{(u)} \leq \frac{2}{3}$$

for $q \geq 3/2$. Thus, the pressure-gradient ∇p_ν has

$$\sigma_q^{(\nabla p)} = \sigma_q^{(p)} - 1 \leq -\frac{1}{3} < 0$$

$\Longrightarrow$ pressure-gradient is dissipation-range.

(c) <u>All</u> of these are <u>inertial-range</u>. In fact,

$$\overline{f}_{\nu,\ell}(\mathbf{x}) = \int d^d r \, G_\ell(\mathbf{r}) f_\nu(\mathbf{x}+\mathbf{r})$$
$$\xrightarrow[\nu\to 0]{} \int d^d r \, G_\ell(\mathbf{r}) f_*(\mathbf{x}+\mathbf{r}) = \overline{f}_{*\ell}(\mathbf{x})$$

which is a smooth function of $\mathbf{x}$, even if f_* is only a distribution.

(d) Since

$$f'_{\nu,\ell} = f_\nu - \overline{f}_{\nu,\ell}$$
$$\xrightarrow[\nu\to 0]{} f_* - \overline{f}_{*,\ell} ,$$

$f'_{\nu,\ell}$ converges as $\nu \to 0$ to an ordinary function $f'_{*,\ell}$ if and only if f_ν does so. Thus, the answers in (d) are the same as those for (b)

$\mathbf{u}'_\ell$ inertial-range

$\nabla \mathbf{u}'_\ell$ dissipation-range

p'_ℓ inertial-range

$\nabla p'_\ell$ dissipation-range

Problem 3. (a) After integrating over space domain Ω, the term $\nabla\cdot J_\ell^{k'}$ vanishes because of the condition that no energy flows across the boundary $\partial\Omega$. Thus,

$$\frac{d}{dt}\int d^dx\, k_\ell = \int d^dx\,[\Pi_\ell - \varepsilon_\ell' + \varphi_\ell'].$$

Integrating in time over the interval $t\in[0,T]$ then gives

$$\int d^dx\, k_\ell(T) - \int d^dx\, k_\ell(0)$$
$$= \int_0^T dt \int d^dx\,[\Pi_\ell - \varepsilon_\ell' + \varphi_\ell'].$$

On the other hand, $\varepsilon_\ell' = \nu\overline{(|\nabla v|^2)}_\ell - \nu|\nabla\bar{v}_\ell|^2$, so that

$$\int d^dx\,\varepsilon_\ell' = \nu\int d^dx\,|\nabla v|^2 - \nu\int d^dx\,|\nabla\bar{v}_\ell|^2$$

since $\int d^dx\,\bar{f}_\ell = \int d^dx\, f$ for any function f. Substituting into the precious equality and rearranging terms gives

$$\nu\int_0^T dt\int d^dx\,|\nabla v|^2 = \int_0^T dt\int d^dx\,[\Pi_\ell + \nu|\nabla\bar{v}_\ell|^2 + \varphi_\ell']$$
$$- \int d^dx\, k_\ell(t)\Big|_{t=0}^{t=T}.$$

(b) Because of convexity of the function $|\mathbf{v}|^2$ in $\mathbf{v}$, it follows that $|\overline{\mathbf{v}}_\ell|^2 \leq \overline{(|\mathbf{v}|^2)}_\ell$, so that

$$\frac{1}{2}\int d^dx\, |\overline{\mathbf{v}}_\ell|^2 \leq \frac{1}{2}\int d^dx\, \overline{(|\mathbf{v}|^2)}_\ell = \frac{1}{2}\int d^dx\, |\mathbf{v}|^2.$$

Since $k_\ell = \frac{1}{2}\overline{(|\mathbf{v}|^2)}_\ell - \frac{1}{2}|\overline{\mathbf{v}}_\ell|^2$, the last inequality is equivalent to

$$\int d^dx\; k_\ell(t) \geq 0$$

for all $t \in [0,T]$. Applying this inequality to $t=T$ gives

$$\nu\int_0^T dt \int d^dx\, |\nabla \mathbf{v}|^2 \leq \int_0^T dt \int d^dx \left[\Pi_\ell + \nu|\nabla\overline{\mathbf{v}}_\ell|^2 + Q_\ell'\right]$$

$$+ \int d^dx\; k_\ell(0).$$

(c) We now apply the inequalities derived in class and the course notes, such as

$$\left|\frac{1}{|\Omega|}\int d^dx\, \Pi_\ell\right| \leq \|\Pi_\ell\|_1$$

$$\leq \|\Pi_\ell\|_{p/3} \quad \text{for } p \geq 3$$

$$\leq (\text{const.})\,\frac{V^3(t)}{L}\left(\frac{\ell}{L}\right)^{3s-1}$$

$$\frac{\nu}{|\Omega|}\int d^dx\,|\nabla\bar{\mathbf{v}}_\ell|^2 = \nu\|\nabla\bar{\mathbf{v}}_\ell\|_2^2$$

$$\leq \nu\|\nabla\mathbf{v}\|_p^2 \quad \text{for } p\geq 3>2$$

$$\leq (\text{const.})\,\frac{\nu V^2(t)}{\ell^2}\left(\frac{\ell}{L}\right)^{2s}$$

$$= (\text{const.})\,\frac{\nu V^2(t)}{L^2}\left(\frac{\ell}{L}\right)^{2s-2}$$

Since $\varphi_\ell' = T_\ell(\mathbf{f};\mathbf{v}) = \int d^3r\, G_\ell(\mathbf{r})\,\delta\mathbf{f}(\mathbf{r})\cdot\delta\mathbf{v}(\mathbf{r}) - \int d^3r\, G_\ell(\mathbf{r})\,\delta\mathbf{f}(\mathbf{r})\cdot \int d^3r'\, G_\ell(\mathbf{r}')\,\delta\mathbf{v}(\mathbf{r}')$

we may apply the same approach as for $T_\ell = T_\ell(\mathbf{v},\mathbf{v})$ to show

$$\left|\frac{1}{|\Omega|}\int d^3x\,\varphi_\ell'\right| \leq \|\varphi_\ell'\|_1$$

$$\leq \|\varphi_\ell'\|_{p/2} \quad \text{for } p/2\geq 3/2>1$$

$$\leq \int d^3r\, G_\ell(\mathbf{r})\,\|\delta\mathbf{f}(\mathbf{r})\|_p\,\|\delta\mathbf{v}(\mathbf{r})\|_p$$

$$+ \int d^3r\, G_\ell(\mathbf{r})\|\delta\mathbf{f}(\mathbf{r})\|_p \int d^3r'\, G_\ell(\mathbf{r}')\|\delta\mathbf{v}(\mathbf{r}')\|_p$$

$$\leq (\text{const.})\,\frac{V^3(t)}{L}\left(\frac{\ell}{L}\right)^{2s},$$

if one assumes (in a dimensionally correct form) that

$$\| \delta \mathbf{f}(\cdot,t;\mathbf{r}) \|_p \leq \frac{V^2(t)}{L} \left| \frac{\mathbf{r}}{L} \right|^s, \quad |\mathbf{r}| < L.$$

Of course, by the results in class,

$$\frac{1}{|\Omega|} \int d^d x \, k_\ell(0) = V^2(0) \left(\frac{\ell}{L} \right)^{2s}.$$

Putting all of the various estimates together gives

$$\langle \varepsilon \rangle \leq (\text{const.}) \frac{V^3}{L} \left[\left(\frac{\ell}{L} \right)^{3s-1} + \frac{1}{Re} \left(\frac{\ell}{L} \right)^{2s-2} + \left(\frac{\ell}{L} \right)^{2s} \right]$$

$$+ \ (\text{const.}) \frac{V^2}{T} \left(\frac{\ell}{L} \right)^{2s}$$

with $V = \max_{t \in [0,T]} V(t)$ and $Re := VL/\nu$. Since $s < 1$

$$\left(\frac{\ell}{L} \right)^{2s} = \left(\frac{\ell}{L} \right)^{3s-1} \left(\frac{\ell}{L} \right)^{1-s} < \left(\frac{\ell}{L} \right)^{3s-1}$$

for $\ell < L$ and

$$\frac{V^3}{L} \left(\frac{\ell}{L} \right)^{3s-1} + \frac{V^2}{T} \left(\frac{\ell}{L} \right)^{2s} = \frac{V^3}{L} \left(\frac{\ell}{L} \right)^{3s-1} \left[1 + \frac{L}{VT} \left(\frac{\ell}{L} \right)^{1-s} \right]$$

$$\leq \left(1 + \frac{L}{VT} \right) \times \frac{V^3}{L} \left(\frac{\ell}{L} \right)^{3s-1} \qquad \ell < L.$$

Thus, we may absorb the last two terms into the first one and obtain

$$\langle \varepsilon \rangle \leq (\text{const.}) \frac{V^3}{L} \left[\left(\frac{\ell}{L}\right)^{3s-1} + \frac{1}{Re}\left(\frac{\ell}{L}\right)^{2s-2} \right].$$

(d) It is a simple calculus exercise to minimize the above upper bound by differentiating with respect to ℓ, and this yields the minimum (smallest bound) for

$$\ell = C_s \, L \, Re^{-1/(1+s)}$$

with $C_s = \left(\frac{2-2s}{3s-1}\right)^{\frac{1}{s+1}}$. Note that for this choice the two terms in the previous upper bound became of the same order of magnitude and, thus,

$$\langle \varepsilon \rangle \leq (\text{const.}) \frac{V^3}{L} Re^{-(3s-1)/(s+1)}.$$

Clearly, if $s > \frac{1}{3}$, then $\langle \varepsilon \rangle \longrightarrow 0$ as $Re \longrightarrow \infty$, and the rate of viscous energy dissipation must vanish as an inverse power of Re.

(e) If, instead,

$$\langle \varepsilon \rangle \sim (\text{const.}) \frac{V^3}{L} Re^{-\alpha} \quad \text{as } Re \to \infty$$

then one arrives at a contradiction unless

$$\frac{3s-1}{s-1} \leq \alpha$$

$$\Longleftrightarrow \quad s \leq \frac{\alpha+1}{3-\alpha} := s_\alpha .$$

In terms of inertial-range structure-function scaling exponents, this translates into the condition

$$\zeta_p \leq s_\alpha p \quad \text{for } p \geq 3 .$$

In a laminar flow, $\langle \varepsilon \rangle = \frac{(\text{const.})}{Re}$ but if $\langle \varepsilon \rangle$ vanishes just a bit slower than this, the previous argument shows that the Navier-Stokes velocity cannot have smoothness $s > s_\alpha \doteq 1 + (\alpha - 1) = \alpha$ for $1 - \alpha \ll 1$, holding uniformly in viscosity. On the other hand, if $\alpha \ll 1$ and the energy dissipation rate is nearly independent of Re, then the critical smoothness exponent $s_\alpha \doteq \frac{1}{3} + \frac{4\alpha}{9}$, which is just slightly larger than Onsager's exponent $1/3$. The conclusions of the argument are thus very robust and do not depend upon exact independence of viscosity.

Problem 4. (a) The result

$$\widetilde{\tau}_\ell(u,u) = \overline{(\tau_\delta(u,u))}_\ell + \tau_\ell(\overline{u}_\delta, \overline{u}_\delta)$$

is just the Germano identity.

(b) In general, $\tau_\ell(u,u)$ denotes the stress obtained from velocity modes in u at scales $< \ell$. Thus, $\tau_\ell(\overline{u}_\delta, \overline{u}_\delta)$ denotes the stress obtained from velocity modes in $\overline{u}_\delta$ at scales $< \ell$. However, $\overline{u}_\delta$ contains only scales $> \delta$. Thus, $\tau_\ell(\overline{u}_\delta, \overline{u}_\delta)$ can be interpreted as the stress at scale ℓ from the velocity modes in the range between δ and ℓ. Note that $\tau_\ell(\overline{u}_\delta, \overline{u}_\delta)$ is a positive matrix, as a stress should be. Next, note that $\tau_\delta(u,u)$ is the stress at length-scale δ from the modes at scales $< \delta$. Thus, the filtered quantity $\overline{(\tau_\delta(u,u))}_\ell$ is the stress at scales ℓ from the velocity modes at scales $< \delta$. Again, $\overline{(\tau_\delta(u,u))}_\ell$ is a positive matrix, as it should be.

(c) If u is Hölder continuous with exponent h, then

$$\tau_\delta(u,u) = O(\delta^{2h}),$$

as shown in class. Since additional filtering will not change this, i.e.

$$|\overline{f}_\ell(x)| \leq \int d^dr\, G_\ell(r)\,|f(x+r)| \leq \|f\|_\infty,$$

it follows at once that also

$$\overline{(\tau_\delta(u,u))}_\ell = O(\delta^{2h}).$$

Problem 5. (a) By general results for $T_\ell(f,g)$

$$f_\ell^\nu = \int d^d r\, G_\ell(r)\, \delta u(r) \times \delta \omega(r)$$
$$- \int d^d r\, G_\ell(r) \delta u(r) \times \int d^d r\, G_\ell(r)\, \delta\omega(r).$$

(b) If G has compact support, then

$$|f_\ell^\nu| \leq \int d^d r\, G_\ell(r)\, |\delta u(r)| \cdot |\delta\omega(r)|$$
$$+ \int d^d r\, G_\ell(r)\, |\delta u(r)| \cdot \int d^d r'\, G_\ell(r')\, |\delta \omega(r')|$$

$$\leq 2\, \delta u(\ell)\, \delta \omega(\ell)$$

with $\delta u(\ell) = \sup_{r<\ell} |\delta u(r)|$, $\delta\omega(\ell) = \sup_{r<\ell} |\delta\omega(r)|$. However, ω is not Hölder continuous in the limit $\nu \to 0$, so we might as well take

$$|\delta\omega(r)| = |\omega(x+r) - \omega(x)|$$
$$\leq |\omega(x+r)| + |\omega(x)| \leq 2\|\omega\|_\infty.$$

Thus,

$$|f_\ell^\nu| \leq 4\|\omega\|_\infty\, \delta u(\ell).$$

(c) The bound is probably <u>not</u> optimal, since u is inertial-range but ω is dissipation-range. Thus, large cancellations should occur in the integral over r.

The correlation coefficient can be estimated by a ratio of time-scales:

$$\rho(\delta\omega(\ell), \delta u(\ell)) \cong \frac{\delta u(\ell)/\ell}{\omega_{max}} \ll 1$$

Thus, a better estimate should be — heuristically —

$$|f^v_\ell| = O^*\left(\delta u(\ell)\,\omega_{max}\cdot\frac{\delta u(\ell)/\ell}{\omega_{max}}\right)$$

$$= O^*\left(\frac{\delta u^2(\ell)}{\ell}\right).$$

Note that this is the same estimate which was obtained (rigorously!) for the subgrid force f^s_ℓ. In fact, using

$$f^v_\ell = f^s_\ell + \nabla k_\ell \quad , \quad k_\ell = \frac{1}{2}\mathrm{tr}\,\tau_\ell(u,u)$$

one can indeed show rigorously that

$$|f^v_\ell| = O\left(\frac{\delta u^2(\ell)}{\ell}\right).$$

Here, we have used also the result

$$\partial_i k_\ell = \frac{1}{2\ell}\Big\{\int d^dr\,(\partial_i G)_\ell(r)\,|\delta u(r)|^2 - 2\int d^dr\,(\partial_i G)_\ell(r)\,\delta u(r)\cdot\int d^dr'\,G_\ell(r')\,\delta u(r')\Big\}$$

$$= O\left(\frac{\delta u^2(\ell)}{\ell}\right).$$

Thus, the heuristic argument about cancellations leads in this case to a result which can be rigorously verified!

(d) Let us first examine the issue of scale locality using the formula

$$f_\ell^v = \int d^d r\, G_\ell(\mathbf{r})\, \delta \mathbf{u}(\mathbf{r}) \times \delta \boldsymbol{\omega}(\mathbf{r})$$
$$- \int d^d r\, G_\ell(\mathbf{r})\, \delta \mathbf{u}(\mathbf{r}) \times \int d^d r'\, G_\ell(\mathbf{r}')\, \delta \boldsymbol{\omega}(\mathbf{r}')$$

We already know that $\delta \mathbf{u}(\mathbf{r})$ is both IR and UV local when $\mathbf{u}$ is Hölder continuous with an exponent $0<h<1$. What about $\delta\boldsymbol{\omega}(\mathbf{r})$? It is certainly IR-local, since

$$\delta\overline{\omega}_\Delta(\mathbf{r}) = \mathbf{r}\cdot\nabla\overline{\omega}_\Delta(x_*) = O(r\Delta^{h-1})$$

and this goes to zero for $\Delta/r \gg 1$, when $h<1$. However, the vorticity increment is <u>not</u> UV-local, since

$$|\delta\omega_\delta'(\mathbf{r})| = |\omega_\delta'(x+\mathbf{r}) - \omega_\delta'(x)| \cong 2\omega_{max}$$

does not need to vanish at all for $\delta \ll r$. Thus, we can draw <u>no conclusion</u> about UV locality of f_ℓ^v. Although $\delta\omega(\mathbf{r})$ is not UV local, there may be sufficient cancellations in the integral over $\mathbf{r}$ to ensure UV-locality.

In fact, this can be proved using

$$f_\ell^v(\mathbf{u},\mathbf{u}) = f_\ell^s(\mathbf{u},\mathbf{u}) + \nabla k_\ell(\mathbf{u},\mathbf{u}),$$

from which estimates such as

$$f_\ell^v(\mathbf{u},\mathbf{u}_\delta') = O\left(\frac{\delta u(\ell)}{\ell}\,\delta u(\delta)\right) = O(\ell^{h-1}\delta^h)$$

can be derived and this goes to zero as $\delta\to 0$, if $h>0$.