Problem 1. (a) By definition
\[ T(u_i, u_j, u_k) = \overline{u_i} u_j u_k - \overline{u_i} \overline{T(u_j, u_k)} - \overline{u_j} \overline{T(u_i, u_k)} - \overline{u_k} \overline{T(u_i, u_j)} - \overline{u_i} \overline{u_j} \overline{u_k} \]
\[ = \overline{u_i} u_j u_k - \overline{u_i} \left( \overline{u_j} u_k - \overline{u_k} u_j \right) - \overline{u_j} \left( \overline{u_i} u_k - \overline{u_k} u_i \right) \]
\[ - \overline{u_k} \left( \overline{u_i} u_j - \overline{u_j} u_i \right) - \overline{u_i} \overline{u_j} \overline{u_k} \]
\[ = \overline{u_i} u_j u_k - \overline{u_i} \overline{u_j} u_k - \overline{u_j} \overline{u_k} u_i - \overline{u_k} \overline{u_i} u_j + \overline{u_i} \overline{u_j} \overline{u_k}. \]

(b) Recall that \( W(\alpha) = \ln Z(\alpha) \) with
\[ Z(\alpha) = \left( \exp(\alpha \cdot u) \right)_L. \]
Thus,
\[ \overline{u_i}(\alpha) = \frac{\partial}{\partial \alpha_i} W(\alpha) = \frac{1}{Z(\alpha)} \frac{\partial Z(\alpha)}{\partial \alpha_i} = \frac{1}{Z(\alpha)} \left( u_i e^{\alpha \cdot u} \right)_L. \]

Similarly,
\[ T_{ij}(\alpha) = \frac{\partial^2}{\partial \alpha_i \partial \alpha_j} W(\alpha) \]
\[ = \frac{\partial}{\partial \alpha_j} \left[ \frac{1}{Z(\alpha)} \left( u_i e^{\alpha \cdot u} \right)_L \right] \]
\[ = \frac{1}{Z(\alpha)} \left( u_i u_j e^{\alpha \cdot u} \right)_L - \frac{1}{Z^2(\alpha)} \left( u_i e^{\alpha \cdot u} \right)_L \left( u_j e^{\alpha \cdot u} \right)_L. \]
and
\[
\tau_{ijk}(\omega) = \frac{\partial^3}{\partial \omega_i \partial \omega_j \partial \omega_k} W(\omega)
\]
\[
= \frac{2}{\partial \omega_k} \left[ \frac{1}{Z(\omega)} (u_i u_j e^{\omega_u})^L - \frac{1}{Z^2(\omega)} (u_i e^{\omega_u})_L (u_j e^{\omega_u})_L \right]
\]
\[
= \frac{1}{Z(\omega)} \left( u_i u_j u_k e^{\omega_u} \right)_L - \frac{1}{Z^2(\omega)} (u_k e^{\omega_u})_L (u_i u_j e^{\omega_u})_L
\]
\[
- \frac{1}{Z^2(\omega)} (u_i u_j e^{\omega_u})_L (u_k e^{\omega_u})_L - \frac{1}{Z^2(\omega)} (u_i e^{\omega_u})_L (u_j u_k e^{\omega_u})_L
\]
\[
+ \frac{2}{Z^3(\omega)} (u_i e^{\omega_u})_L (u_j e^{\omega_u})_L (u_k e^{\omega_u})_L.
\]

Finally, setting \( \omega = 0 \), we get – up to (since \( Z(0) = 1 \))

\[
\tau(\phi_i \phi_j, \phi_k) = \tau_{ijk}(\omega) \bigg|_{\omega = 0}
\]
\[
= u_i u_j u_k - u_k u_i u_j - u_i u_j u_k - u_j u_k u_i + 2u_i u_j u_k .
\]

(c) In Homework #1, Problem 2(b) it was shown that 3rd-order cumulants are always shift-invariant. Since

\[
\tau_G(\phi_i, \phi_j, \phi_k) = \langle \phi_i(\cdot) \phi_j(\cdot) \phi_k(\cdot) \rangle_G
\]

Employing the shift \( \phi_i(x; \omega) \rightarrow \phi_i(x; \omega) - \phi_i(\cdot) = \delta_i(x; \omega) \),

\[
\tau_G(\phi_i, \phi_j, \phi_k) = \langle \delta_i(\cdot) \delta_j(\cdot) \phi_k(\cdot) \rangle_G
\]
\[
= \langle \delta_i(\cdot) \delta_j(\cdot) \delta_k(\cdot) \rangle_G - \langle \delta_i(\cdot) \rangle_G \langle \delta_j(\cdot) \delta_k(\cdot) \rangle_G - \langle \delta_j(\cdot) \rangle_G \langle \delta_i(\cdot) \delta_k(\cdot) \rangle_G
\]
\[
- \langle \delta_k(\cdot) \rangle_G \langle \delta_i(\cdot) \delta_j(\cdot) \rangle_G + 2 \langle \delta_i(\cdot) \rangle_G \langle \delta_j(\cdot) \delta_k(\cdot) \rangle_G .
\]
Problem 2, (a) Using the relation

$$ \int_0^\infty E_{F_k}(k) \, dk = \langle |F_k|^2 \rangle $$

we see that $$ \langle |F_k|^2 \rangle \to +\infty$$ if \( E_{F_k}(k) \sim k^n \) for \( n > 1 \), but that
$$ \langle |F_k|^2 \rangle = +\infty$$ if \( E_{F_k}(k) \sim k^n \) for \( n \leq 1 \). Thus, \( F \) is inexact-range if \( n > 1 \) and dissipation-range if \( n \leq 1 \).

(b) Since
$$ \langle |U_\nu|^2 \rangle \to +\infty$$
(certainly in decaying turbulence), \( U_\nu \) is inexact-range. However,
$$ \langle |\nabla U_\nu|^2 \rangle = \frac{\varepsilon}{\nu} \to \infty$$
as \( \nu \to 0 \), so that \( (\nabla U_\nu) \) is dissipation-range. Since the pressure has

$$ \| \delta p(x) \|_q \leq \text{const.} \| \delta U_\nu(x) \|_2^2 = O(\frac{\varepsilon^{(m)}}{\nu^{2q} / 2q})$$
it follows that \( \delta p \) is even more regular than \( \delta U_\nu \). Thus, pressure \( p_\nu \) is inexact-range. However, the evidence is that \( \varepsilon^{(m)} \leq \frac{2}{3} q \), \( q \geq 3/2 \)

(account for energy dissipation) so that
$$ \sigma(p) = \frac{\varepsilon^{(m)}}{q} \approx \sigma^{(m)} \leq \frac{2}{3} q \leq \frac{2}{3}$$

for \( q \geq 3/2 \). Thus, the pressure-gradient \( \nabla p_\nu \) has
$$ \sigma^{(\nabla p)} = \sigma^{(p)} - 1 \leq -\frac{1}{3} < 0$$
$$ \Rightarrow \text{pressure-gradient is dissipation-range}.$$
(c) All of these are inertial-range. In fact,

\[ f_{\nu, l}^l (x) = \int dx \, \mathbb{E}(r) - f_{\nu}(x+r) \]

\[ \lim_{\nu \to 0} \int dx \, \mathbb{E}(r) - f_{\nu}(x+r) = f_{\nu, l}^l (x) \]

which is a smooth function of \( x \), even if \( f^l \) is only a distribution.

(d) Since

\[ f_{\nu, l}^l = f_{\nu} - \bar{f}_{\nu, l} \]

\[ \lim_{\nu \to 0} f_{\nu} - \bar{f}_{\nu, l} \]

\( f_{\nu, l}^l \) converges as \( \nu \to 0 \) to an ordinary function \( f_{\nu, l}^l \) if and only if \( f_{\nu} \) does so. Thus, the answers in (d) are the same as those for (c).

\[ u_{\nu}^l \quad \text{inertial-range} \]

\[ \nabla u_{\nu}^l \quad \text{dissipation-range} \]

\[ p_{\nu}^l \quad \text{inertial-range} \]

\[ \nabla p_{\nu}^l \quad \text{dissipation-range} \]
Problem 3. (a) After integrating over space domain $\Omega$, the term $\nabla \cdot J^1_k$ vanishes because of the condition that no energy flows across the boundary $\partial \Omega$. Thus,

$$\frac{d}{dt} \int d^d x \ k_e = \int d^d x \left[ T^1_e - \varepsilon'_e + \Phi'_e \right].$$

Integrating in time over the interval $t \in [0, T]$ then gives

$$\int d^d x \ k_e(T) - \int d^d x \ k_e(0) = \int_0^T \int d^d x \left[ T^1_e - \varepsilon'_e + \Phi'_e \right].$$

On the other hand, $\varepsilon'_e = \sqrt{(\nabla v)^2} - \nu \left| \nabla \varepsilon_e \right|^2$, so that

$$\int d^d x \ \varepsilon'_e = \nu \int d^d x \ \left| \nabla v \right|^2 - \nu \int d^d x \ \left| \nabla \varepsilon_e \right|^2$$

since $\int d^d x \ f_e = \int d^d x \ f \nu$ for any function $f$. Substituting into the previous equality and reordering terms gives

$$\nu \int_0^T \int d^d x \ \left| \nabla v \right|^2 = \int d^d x \left[ T^1_e + \nu \left| \nabla \varepsilon_e \right|^2 + \Phi'_e \right]_{t=T} - \int d^d x \ k_e(t) \bigg|_{t=0}. $$
(b) Because of convexity of the function $|v|^2$ in $v$, it follows that $|v_e|^2 \leq (|v|^2)_e$, so that

$$\frac{1}{2} \int d^d x |v_e|^2 \leq \frac{1}{2} \int d^d x (|v|^2)_e = \frac{1}{2} \int d^d x |v|^2.$$

Since $k_e = \frac{1}{2} (|v|^2)_e - \frac{1}{2} |v_e|^2$, the last inequality is equivalent to

$$\int d^d x k_e (t) \geq 0$$

for all $t \in [0, T]$. Applying this inequality to $t = T$ gives

$$\nu \int_0^T dt \int d^d x |\nabla v|^2 \leq \int_0^T dt \int d^d x \left[ T_I e + \nu |\nabla v_e|^2 + Q_e \right]$$

$$\quad + \int d^d x k_e (0).$$

(c) We now apply the inequalities derived in class and the course notes, such as

$$\left| \frac{1}{|x|} \int d^d x T_I e \right| \leq \| T_I e \|_1,$$

$$\leq \| T_I e \|_{p,13} \quad \text{for } p \geq 3$$

$$\leq (\text{const.}) \frac{V^3(\epsilon)}{L} \left( \frac{\epsilon}{L} \right)^{35-1}$$
\[
\frac{\nu}{|\Omega|} \int_{\Omega} |\nabla v_2|^2 = \nu \| \nabla v_2 \|_2^2 \\
\leq \nu \| \nabla v \|_p^2 \quad \text{for } p > 3 > 2 \\
\leq (\text{const.}) \frac{\nu V^2(t)}{L^2} \left( \frac{L}{L} \right)^{2s} \\
= (\text{const.}) \frac{\nu V^2(t)}{L^2} \left( \frac{L}{L} \right)^{2s-2}
\]

Since \( Q'_2 = T_2(f; v) = \int_{\Omega} g_2(r) \delta f(r) \delta v(r) - \int_{\Omega} g_2(r) \delta f(r) \),
we may apply the same approach as for \( T_2 = T_2(v, v) \) to show

\[
\left| \frac{1}{|\Omega|} \int_{\Omega} Q'_2 \right| \leq \| Q'_2 \|_1 \\
\leq \| Q'_2 \|_{p/2} \quad \text{for } p/2 > 3/2 > 1 \\
\leq \int_{\Omega} g_2(r) \| \delta f(r) \|_p \| \delta v(r) \|_p \\
+ \int_{\Omega} g_2(r) \| \delta f(r) \|_p \int_{\Omega} g_2(r') \| \delta v(r') \|_p \\
\leq (\text{const.}) \frac{V^2(t)}{L} \left( \frac{L}{L} \right)^{2s}
\]
if one assumes (in a dimensionally correct form) that
\[ \| \delta f(\cdot, t; \tau) \|_p \leq \frac{V^2(t)}{L} \frac{|\tau|}{L}^s, \quad |\tau| < L. \]

Of course, by the results in class,
\[ \frac{1}{|L|} \int d^d x \langle x | \zeta \rangle = V^2(x) \left( \frac{L}{L} \right)^2. \]

Putting all of the various estimates together gives
\[ \langle \varepsilon \rangle \leq \text{const.} \frac{V^3}{L} \left[ \left( \frac{L}{L} \right)^{3s-1} + \frac{1}{\text{Re}} \left( \frac{L}{L} \right)^{2s-2} + \left( \frac{L}{L} \right)^{2s} \right] \]
\[ + \text{const.} \frac{V^2}{1} \left( \frac{L}{L} \right)^{2s} \]

with \( V = \max_{t \in [0, T]} V(t) \) and \( \text{Re} := VL/\nu \). Since \( s < 1 \)
\[ \left( \frac{L}{L} \right)^{2s} = \left( \frac{L}{L} \right)^{3s-1} \left( \frac{L}{L} \right)^{1-s} < \left( \frac{L}{L} \right)^{3s-1} \]
for \( \ell < L \) and
\[ \frac{V^3}{L} \left( \frac{L}{L} \right)^{3s-1} + \frac{V^2}{1} \left( \frac{L}{L} \right)^{2s} = \frac{V^3}{L} \left( \frac{L}{L} \right)^{3s-1} \left[ 1 + \frac{L}{\nu V T} \left( \frac{L}{L} \right)^{1-s} \right] \]
\[ \leq \left( 1 + \frac{L}{\nu V T} \right) \times \frac{V^3}{L} \left( \frac{L}{L} \right)^{3s-1}, \quad \ell < L. \]
Thus, we may absorb the last two terms into the first one and obtain

$$\langle \varepsilon \rangle \leq \text{(const.)} \frac{V^3}{L} \left[ \left( \frac{L}{L} \right)^{3s-1} + \frac{1}{Re} \left( \frac{L}{L} \right)^{2s-2} \right].$$

(d) It is a simple calculus exercise to minimize the above upper bound by differentiating with respect to \( \lambda \), and this yields the minimum (smallest bound) for

$$\lambda = C_5 L Re^{-\frac{1}{1+(s)^{1}}}$$

with \( C_5 = \left( \frac{2-2s}{3s-1} \right)^{\frac{1}{5+1}} \). Note that for this choice the two terms in the previous upper bound became of the same order of magnitude and, thus,

$$\langle \varepsilon \rangle \leq \text{const.} \frac{V^3}{L} Re^{-\frac{(3s-1)/(5+1)}}.$$  

Clearly, if \( s > \frac{1}{3} \), then \( \langle \varepsilon \rangle \to 0 \) as \( Re \to \infty \), and the rate of viscous energy dissipation must vanish as an inverse power of \( Re \).
(e) If, instead,

\[
\langle \xi \rangle \sim (\text{const.}) \frac{V^3}{L} \text{ Re}^{-\alpha} \quad \text{as } \text{ Re} \to \infty
\]

then one arrives at a contradiction unless

\[
\frac{3s-1}{s-1} \leq \alpha
\]

\[
\iff s \leq \frac{\alpha+1}{3-\alpha} := S_\alpha.
\]

In terms of inertial-range structure-function scaling exponents, this translates into the condition

\[
S_p \leq S_\alpha \quad \text{for } p \geq 3.
\]

In a laminar flow, \( \langle \xi \rangle = \frac{(\text{const.})}{\text{Re}} \) but if \( \langle \xi \rangle \) vanishes just a bit slower than this, the previous argument shows that the Navier-Stokes velocity cannot have smoothness \( s > S_\alpha \equiv 1 + (\alpha - 1) = \alpha \) for \( 1 - \alpha \ll 1 \), holding uniformly in viscosity. On the other hand, if \( \alpha \ll 1 \) and the energy dissipation rate is nearly independent of \( \text{Re} \), then the critical smoothness exponent \( S_\alpha \equiv \frac{1}{3} + \frac{4\alpha}{9} \), which is just slightly larger than Onsager's exponent \( 1/3 \). The conclusions of the argument are thus very robust and do not depend upon exact independence of viscosity.
Problem 4. (a) The result

\[ \hat{\tau}(u,u) = (T_\delta(u,u))_\ell + \tau_\delta(u,\delta, u, \delta) \]

is just the geometric identity.

(b) In general, \( T_\delta(u,u) \) denotes the stress obtained from velocity modes in \( u \) at scales \(< \ell \). Thus, \( \tau_\delta(u,\delta, u, \delta) \) denotes the stress obtained from velocity modes in \( u_\delta \) at scales \(< \ell \). However, \( u_\delta \) contains only scales \( > \delta \). Thus, \( \tau_\delta(u,\delta, u, \delta) \) can be interpreted as the stress at scale \( \ell \) from the velocity modes in the range between \( \delta \) and \( \ell \). Note that \( \tau_\delta(u,\delta, u, \delta) \) is a positive matrix, as a stress should be. Next, note that \( T_\delta(u,u) \) is the stress at length-scale \( \delta \) from the modes at scales \( < \delta \).

Thus, the filtered quantity \((T_\delta(u,u))_\ell\) is the stress at scales \( \ell \) from the velocity modes at scales \( < \delta \). Again, \((T_\delta(u,u))_\ell\) is a positive matrix, as it should be.

(c) If \( u \) is Hölder continuous with exponent \( H \), then

\[ T_\delta(u,u) = O(\delta^{2H}) \]

as shown in class. Since additional filtering will not change this, i.e.

\[ |\hat{T}_\ell| \leq \int |\xi(r)| |\hat{\tau}(r)| \leq \|\hat{\tau}\|_{L^1} \leq \|\tau\|_{L^1} \leq \|\tau\|_{L^\infty} \leq \|u\|_{C^H} \]

it follows at once that also

\[ (T_\delta(u,u))_\ell = O(\delta^{2H}) \]
Problem 5. (a) By general results for $T_2(f, g, \nu)$

\[ f'_{2} = \int d^{4}r \, G_{2}(r) \delta u(r) \times \delta w(r) \]
\[ - \int d^{4}r \, G_{2}(r) \delta u(r) \times \int d^{4}r' \, G_{2}(r') \delta w(r') \].

(b) If $G$ has compact support, then

\[ |f'_{2}| \leq \int d^{4}r \, G_{2}(r) \left( |\delta u(r)| \cdot |\delta w(r)| \right) \]
\[ + \int d^{4}r \, G_{2}(r) \left( |\delta u(r)| \cdot \int d^{4}r' \, G_{2}(r') \left| \delta w(r') \right| \right) \]
\[ \leq 2 \delta u(l) \delta w(l) \]

with $\delta u(l) = \sup_{r \leq l} |\delta u(r)|$, $\delta w(l) = \sup_{r \leq l} |\delta w(r)|$. However, $w$ is not Hölder continuous in the limit $\nu \to 0$, so we might as well take

\[ |\delta w(r)| = |w(x+r) - w(x)| \]
\[ \leq |w(x+r)| + |w(x)| \leq 2 \| w \|_{\infty}. \]

Thus,

\[ |f'_{2}| \leq 4 \| w \|_{\infty} \delta u(l). \]

(c) The bound is probably not optimal, since $w$ is inertial-range but $w$ is dissipation-range. Thus, large cancellations should occur in the integral over $r$. 
The correlation coefficient can be estimated by a ratio of time-scales:

\[ \rho(\delta u_k, \delta u_k) \approx \frac{\delta u_k(\ell)/\ell}{\omega_{\max}} < 1 \]

Thus, a better estimate should be—heuristically—

\[ |f_{\ell}^v| = O\left( \delta u_k(\ell) \omega_{\max} \cdot \frac{\delta u_k(\ell)/\ell}{\omega_{\max}} \right) \]

\[ = O\left( \frac{\delta u_k(\ell)}{\ell} \right) \]

Note that this is the same estimate which was obtained (rigorously!) for the subgrid force \( f_{\ell}^s \). In fact, using

\[ f_{\ell}^v = f_{\ell}^s + \nabla k_{\ell} \quad , \quad k_{\ell} = \frac{1}{2} \text{tr} \mathcal{T}_{\ell}(u, u) \]

one can indeed show rigorously that

\[ |f_{\ell}^v| = O\left( \frac{\delta u_k(\ell)}{\ell} \right) \]

Here, we have used also the result

\[ \partial_t k_{\ell} = \frac{1}{2\ell} \left\{ \int d^d r \left( \partial_i \mathcal{G}_{\ell}(r) \partial_j \delta u(r) \right)^2 \right. \]

\[ - 2 \int d^d r \left( \partial_i \mathcal{G}_{\ell}(r) \delta u(r) \right) \left. \cdot \int d^d r' \mathcal{G}_{\ell}(r') \delta u(r') \right\} \]

\[ = O\left( \frac{\delta u_k(\ell)}{\ell} \right) \]

Thus, the heuristic argument about cancellations leads us to a result which can be rigorously verified!
(d) Let us first examine the issue of scale locality using the formula
\[ f^v_\delta = \int \int_\delta G_\delta(r) \delta u(r) \times \delta w(r) \]
\[ - \int \int_\delta G_\delta(r) \delta u(r) \times \int \int_\delta G_\delta(r') \delta w(r') \]

We already knew that \( \delta w(r) \) is both IR and UV local when \( u \) is Hölder continuous with an exponent \( 0 < h < 1 \). What about \( \delta w(r) \)? It is certainly IR-local, since
\[ \delta w_A(r) = r \cdot \nabla \delta w_A(x) = O(r A^{h-1}) \]
and this goes to zero for \( A/r \gg 1 \), when \( h < 1 \). However, the vorticity increment is \underline{not} UV-local, since
\[ |\delta w^s_\delta(r)| = |w^s_\delta(x+r) - w^s_\delta(x)| \leq 2 \omega_{\text{max}} \]
does not need to vanish at all for \( \delta \ll r \). Thus, we can draw no conclusion about UV locality of \( f^v_\delta \). Although \( \delta w(r) \) is not UV local, there may be sufficient cancellations in the integral over \( r \) to ensure UV-locality.

In fact, this can be proved using
\[ f^v_\delta(u, u) = f^s_\delta(u, u) + \nabla k^s_\delta(u, u) \]
from which estimates such as
\[ f^v_\delta(u, u_\delta) = O(\frac{\delta u(\delta)}{\delta u(L)}) = O(L^{h-1} \delta^h) \]
can be derived and this goes to zero as \( \delta \to 0 \), if \( h > 0 \).