Homework No.6, 553.793, Due March 28, 2025.

1. (a) Use the definition of the 3rd-order cumulant to show that

$$\tau(v_i, v_j, v_k) = \overline{v_i v_j v_k} - \overline{u}_i \overline{v_j v_k} - \overline{u}_j \overline{v_i v_k} - \overline{u}_k \overline{v_i v_j} + 2\overline{u}_i \overline{u}_j \overline{u}_k.$$

(b) Derive the same formula by using the generating function:

$$\tau(v_i, v_j, v_k) = \frac{\partial^3}{\partial \alpha_i \partial \alpha_j \partial \alpha_k} W(\boldsymbol{\alpha}) \Big|_{\boldsymbol{\alpha}=\boldsymbol{o}}.$$

(c) Use the result of Homework 4, Problem 2(b) to argue that

$$\tau(v_i, v_j, v_k) = \langle \delta v_i \delta v_j \delta v_k \rangle - \langle \delta v_i \rangle \langle \delta v_j \delta v_k \rangle - \langle \delta v_j \rangle \langle \delta v_i \delta v_k \rangle - \langle \delta v_k \rangle \langle \delta v_i \delta v_j \rangle + 2 \langle \delta v_i \rangle \langle \delta v_j \rangle \langle \delta v_k \rangle.$$

2. We may give a formal mathematical definition of an *inertial range variable* f_ν as one whose zero-viscosity limit $f_* = \lim_{\nu \to 0} f_\nu$ is an ordinary function, not a distribution. Thus, the variable is inertial-range if f_* is in the Besov space B^s_p for some $s > 0$ and $p \geq 1$, i.e. satisfies

$$\|f_*\|_p < \infty, \quad \|\delta f(\mathbf{r})\|_p = O(|\mathbf{r}|^s),$$

or even if f_* is in L_p for some $p \geq 1$. However, if the limit f_* exists only as a distribution, then the variable is called *dissipation range*. For example, if f_* belongs only to B^s_p for some negative $s < 0$, then it is dissipation-range.

(a) In terms of the spectral exponent n that appears in the scaling relation,

$$E_{f_*}(k) \sim (\text{const.}) k^{-n},$$

for what n is the variable f_ν inertial range and for what n dissipation range?

State which of the following variables are inertial-range and which dissipation range. Explain your answers:

(b) velocity $\mathbf{v}$, velocity-gradient $\boldsymbol{\nabla}\mathbf{v}$, pressure p, pressure-gradient $\boldsymbol{\nabla} p$.

(c) large-scale velocity $\overline{\mathbf{v}}_\ell$, large-scale velocity-gradient $\boldsymbol{\nabla}\overline{\mathbf{v}}_\ell$, large-scale pressure $\overline{p}_\ell$, large-scale pressure-gradient $\boldsymbol{\nabla}\overline{p}_\ell$.

(d) small-scale velocity $\mathbf{u}'_\ell$, small-scale velocity-gradient $\boldsymbol{\nabla}\mathbf{u}'_\ell$, small-scale pressure p'_ℓ, small-scale pressure-gradient $\boldsymbol{\nabla} p'_\ell$.

3. (Double Points) In this problem we pursue an alternative approach to proving Onsager's result on the singularities required for turbulent anomalous dissipation, based upon the balance equation for unresolved kinetic energy $k_\ell = (1/2)\mathrm{Tr}\,\boldsymbol{\tau}_\ell$, or

$$\partial_t k_\ell + \boldsymbol{\nabla\cdot}\mathbf{J}_\ell^{K\prime} = \Pi_\ell - \varepsilon_\ell' + Q_\ell',$$

with

$$\Pi_\ell = -\boldsymbol{\nabla}\overline{\mathbf{v}}_\ell\boldsymbol{:}\tau_\ell(\mathbf{v},\mathbf{v}), \quad \varepsilon_\ell' = \nu\overline{(|\boldsymbol{\nabla}\mathbf{v}|^2)}_\ell - \nu|\boldsymbol{\nabla}\overline{\mathbf{v}}_\ell|^2, \quad Q_\ell' = \tau_\ell(\mathbf{v};\mathbf{f})$$

(a) Assuming that there is no transport of kinetic energy across the boundary of the spatial flow domain Ω, derive the following balance equation for the space-time integrated viscous dissipation

$$\int_0^T dt \int_\Omega d^dx\ \nu|\boldsymbol{\nabla}\mathbf{v}|^2 = \int_0^T dt \int_\Omega d^dx\left[\Pi_\ell + \nu|\boldsymbol{\nabla}\overline{\mathbf{v}}_\ell|^2 + Q_\ell'\right] - \int d^dx\ k_\ell(t)\Big|_{t=0}^{t=T}.$$

(b) Because of convexity, the kinetic energy observed by a myopic observer with velocity resolved to length-scale ℓ can only be less than the true kinetic energy, or $(1/2)\int|\overline{\mathbf{v}}_\ell(t)|^2 \leq (1/2)\int|\mathbf{v}(t)|^2$. Use this result to show that

$$\int_0^T dt \int_\Omega d^dx\ \nu|\boldsymbol{\nabla}\mathbf{v}|^2 \leq \int_0^T dt \int_\Omega d^dx\left[\Pi_\ell + \nu|\boldsymbol{\nabla}\overline{\mathbf{v}}_\ell|^2 + Q_\ell'\right] + \int d^dx\ k_\ell(0).$$

(c) Now assume that the solution of the Navier-Stokes equation is *Besov regular* with $\mathbf{v}(t) \in B_p^{s,\infty}(\Omega)$ for $0 < s < 1$ and $p \geq 3$, uniformly in viscosity $\nu > 0$ and time $t \in [0,T]$, so that

$$\|\delta\mathbf{v}(\cdot,t;\mathbf{r})\|_p \leq V(t)\left|\frac{\mathbf{r}}{L}\right|^s, \quad |\mathbf{r}| < L$$

with a positive constant $V(t)$ independent of ν and $\max_{t\in[0,T]} V(t) < \infty$. The external body force $\mathbf{f}$ is usually much smoother, but let us assume here only that it is uniformly Besov regular in the same space as the velocity. Using the estimates presented in the lecture notes, derive the following bound on the space-time averaged viscous dissipation

$$\langle\varepsilon\rangle := \frac{1}{T}\int_0^T dt\ \frac{1}{|\Omega|}\int_\Omega d^dx\ \nu|\boldsymbol{\nabla}\mathbf{v}|^2 = O\left(\ell^{3s-1}\right) + O\left(\nu\ell^{2s-2}\right) + O\left(\ell^{2s}\right)$$

where the first term on the right arises from energy flux, the second from resolved viscous dissipation at length-scale ℓ, and the third from forcing and initial conditions. When $s < 1$, show that the third term is smaller than the first for small ℓ and thus the bound on the right is $O\left(\ell^{3s-1}\right) + O\left(\nu\ell^{2s-2}\right)$.

(d) The length-scale ℓ is an arbitrary resolution scale that was introduced to regularize ultraviolet divergences as $\nu \to 0$. It has no intrinsic physical significance and can be chosen at will. Show that the choice $\ell = \nu^{1/((s+1)}$ minimizes the upper bound in part (c) [or, in a dimensionally correct form, $\ell = L \cdot Re^{-1/(s+1)}$ with $Re = VL/\nu$] and the upper bound then becomes

$$\langle \varepsilon \rangle = O\left(\nu^{(3s-1)/(s+1)}\right).$$

Conclude that the viscous kinetic energy dissipation must vanish as $\nu \to 0$ if $s > 1/3$.

(e) Conversely, suppose the viscous energy dissipation vanishes slowly as $\nu \to 0$, i.e.

$$\langle \varepsilon \rangle \propto \nu^{\alpha}, \quad 0 \leq \alpha < 1.$$

Show then that the velocity of the Navier-Stokes solution cannot belong to the Besov space $B_p^{s,\infty}(\Omega)$ uniformly in viscosity, for any $p \geq 3$ and $s > s_\alpha := (\alpha+1)/(3-\alpha)$. If α is very small, then $s_\alpha = 1/3 + \epsilon$ and Onsager's singularity prediction holds even if the energy dissipation rate vanishes as $\nu \to 0$, as long as it does so sufficiently slowly.

4. We consider here a more systematic approach to decompose subscale stress $\tau_\ell(\mathbf{v}, \mathbf{v})$ into one part from length-scales in the range $[\delta, \ell]$, for $\delta < \ell$, and another part from length-scales less than δ.

(a) Define $\widetilde{G}_\ell = G_\ell * G_\delta$. If $\delta < \ell$, then $\widetilde{G}_\ell$ defines a filter kernel at length-scale ℓ. Show that

$$\widetilde{\tau}_\ell(\mathbf{v}, \mathbf{v}) = \overline{(\tau_\delta(\mathbf{v}, \mathbf{v}))_\ell} + \tau_\ell(\overline{\mathbf{v}}_\delta, \overline{\mathbf{v}}_\delta).$$

(b) Explain why $\overline{(\tau_\delta(\mathbf{v}, \mathbf{v}))_\ell}$ can be regarded as the contribution to stress $\widetilde{\tau}_\ell(\mathbf{v}, \mathbf{v})$ from length-scales $< \delta$, while $\tau_\ell(\overline{\mathbf{v}}_\delta, \overline{\mathbf{v}}_\delta)$ is the contribution from scales between δ and ℓ.

(c) If $\mathbf{v}$ is Hölder continuous at point $\mathbf{x}$ with exponent $0 < h < 1$, show that

$$\overline{(\tau_\delta(\mathbf{v}, \mathbf{v}))_\ell} = O(\delta^{2h}).$$

5. The *turbulent vortex-force* is defined by

$$\mathbf{f}_\ell^v = \overline{(\mathbf{v} \times \boldsymbol{\omega})}_\ell - \overline{\mathbf{v}}_\ell \times \overline{\boldsymbol{\omega}}_\ell.$$

(a) Write an expression for $\mathbf{f}_\ell^v$ in terms of velocity increments $\delta\mathbf{v}(\mathbf{r})$ and vorticity increments $\delta\boldsymbol{\omega}(\mathbf{r})$.

(b) Use part (a) to derive a bound $|\mathbf{f}_\ell^v| = O(\|\boldsymbol{\omega}\|_\infty \delta u(\ell))$ for any filter kernel with compact support).

(c) Do you believe that the upper bound in part (b) is optimal? If so, explain. If not, give an argument for a better estimate.

(d) Is the turbulent vortex-force IR scale-local? UV scale-local? Justify your answers.