Homework No.5, 553.793, Due March 14, 2025

1. Tennekes & Lumley, Problem 2.3. We shall discuss this problem in greater detail when we turn to the general subject of Lagrangian dynamics and mixing later in the course. In the meantime, this is a good exercise with which to hone your physical intuition about inertial-range dynamics!

2. The *Taylor microscale* λ was introduced by G. I. Taylor, "Statistical theory of turbulence," Proc. Roy. Soc. Lond. A **151** 421-444 (1935). It is conventionally defined by the relation

$$\varepsilon = 15\nu u_0^2/\lambda^2$$

where ε is mean energy dissipation rate (per mass) and u_0 is the rms value of a single velocity component u. The factor of 15 arises from the relation for isotropic 3D turbulence that $2\langle S_{ij}S_{ij}\rangle = 15\langle(\partial u/\partial x)^2\rangle$, so that $\lambda^2 = \langle u^2\rangle/\langle(\partial u/\partial x)^2\rangle$. The Taylor scale is a kind of "hybrid" length-scale, since it is formed from a combination of an inertial-range quantity $\langle u^2\rangle$ and a dissipation-range quantity $\langle(\partial u/\partial x)^2\rangle$.

(a) Assuming that $\varepsilon \sim u_0^3/L$ for $Re = u_0L/\nu \gg 1$, derive the Reynolds number scaling of the ratio λ/L. Use this result to relate the Reynolds number Re based on the integral length L and the *Taylor-scale Reynolds number* $Re_\lambda = u_0\lambda/\nu$.

(b) Derive the Reynolds number scaling of the ratio η/λ, where $\eta = \nu^{3/4}\varepsilon^{-1/4}$ is the Kolmogorov dissipation length-scale.

(c) Do Tennekes & Lumley, Problem 3.1.

3. Verify directly, without using the "shift trick", the formula discussed in class for the subscale turbulent force:

$$\begin{aligned} f^S_{\ell i} &= -\partial_j \tau_{\ell ij} \\ &= \frac{1}{\ell}\left\{\int d^dr\,(\partial_j G)_\ell(\mathbf{r})\delta u_i(\mathbf{r})\delta u_j(\mathbf{r}) - \int d^dr\,(\partial_j G)_\ell(\mathbf{r})\delta u_i(\mathbf{r})\int d^dr'\,G_\ell(\mathbf{r}')\delta u_j(\mathbf{r}')\right\} \end{aligned}$$

Hint: Substitute $\delta u_i(\mathbf{r};\mathbf{x}) = u_i(\mathbf{x}+\mathbf{r}) - u_i(\mathbf{x})$, etc. and simplify.

4. Verify that coarse-grained pressure $\overline{p}_\ell$ at length-scale ℓ satisfies the equation

$$-\triangle\,\overline{p}_\ell = \frac{1}{\ell^2}\int d^dr\,(\partial_i\partial_j G)_\ell(\mathbf{r})\delta u_i(\mathbf{r})\delta u_j(\mathbf{r}).$$

Hint: Again substitute for $\delta u_i(\mathbf{r};\mathbf{x})$ etc. and use the Poisson equation for pressure.

5. Relatively sophisticated arguments were given in class that $\delta p(\ell) \sim [\delta u(\ell)]^2$, so that $\delta p(\ell) \sim (\varepsilon\ell)^{2/3}$ in K41 theory. Since this is exactly the same result as would be obtained by dimensional analysis, you may cynically and skeptically conclude that all of the fancy mathematics is unnecessary and that dimensional analysis will always give the right answer, or nearly so. However, there are quantities for which K41 dimensional analysis fails badly. One of these is the kinetic energy density $e(\mathbf{x}, t) = (1/2)|\mathbf{u}(\mathbf{x}, t)|^2$ (per mass), which has the same dimensions as kinematic pressure $p(\mathbf{x}, t)$. K41 dimensional analysis would thus lead one to expect that also $\delta e(\ell) \sim (\varepsilon\ell)^{2/3}$. Experimental observations in the atmospheric boundary layer are consistent instead with

$$\delta e(\ell) \sim u_{rms}(\varepsilon\ell)^{1/3}.$$

See van Atta & Wyngaard, "On higher-order spectra of turbulence," J. Fluid Mech. **72** 673–694 (1975).

(a) Show that

$$\delta e(\mathbf{r}; \mathbf{x}) = \mathbf{u}_{av}(\mathbf{x} + \mathbf{r}, \mathbf{x}) \cdot \delta\mathbf{u}(\mathbf{r}; \mathbf{x}),$$

where

$$\mathbf{u}_{av}(\mathbf{x} + \mathbf{r}, \mathbf{x}) = \frac{1}{2}[\mathbf{u}(\mathbf{x} + \mathbf{r}) + \mathbf{u}(\mathbf{x})]$$

is the average of the velocity at the two points $\mathbf{x}$ and $\mathbf{x} + \mathbf{r}$.

(b) Use the identity in (a) to give a plausible explanation of the experimental observations of van Atta & Wyngaard.