Homework #4

Problem 1 (a) By definition

$$\partial_i \bar{f}_\ell(\mathbf{x}) = \lim_{\varepsilon\to 0} \frac{\bar{f}_\ell(\mathbf{x}+\varepsilon\hat{\mathbf{e}}_i) - \bar{f}_\ell(\mathbf{x})}{\varepsilon}$$

$$= \lim_{\varepsilon\to 0} \int d^3r\, G_\ell(\mathbf{r}) \left[\frac{f(\mathbf{x}+\varepsilon\hat{\mathbf{e}}_i+\mathbf{r}) - f(\mathbf{x}+\mathbf{r})}{\varepsilon}\right]$$

using the definition of $\bar{f}_\ell$. Under reasonable assumptions (e.g., if f is uniformly differentiable in a region of radius $\approx \ell$ around $\mathbf{x}$), one can take the limit inside the integral to obtain

$$\partial_i \bar{f}_\ell(\mathbf{x}) = \int d^3r\, G_\ell(\mathbf{r}) \lim_{\varepsilon\to 0}\left[\frac{f(\mathbf{x}+\mathbf{r}+\varepsilon\hat{\mathbf{e}}_i) - f(\mathbf{x}+\mathbf{r})}{\varepsilon}\right]$$

$$= \int d^3r\, G_\ell(\mathbf{r}) (\partial_i f)(\mathbf{x}+\mathbf{r}).$$

(b) Again using the definition of $\bar{f}_\ell$,

$$\int d^dx\, \bar{f}_\ell(\mathbf{x}) = \int d^dx \int d^dr\, G_\ell(\mathbf{r}) f(\mathbf{x}+\mathbf{r})$$

$$= \int d^dx \int d^dx'\, G_\ell(\mathbf{x}-\mathbf{x}') f(\mathbf{x}') \quad w/\ \mathbf{x}' \equiv \mathbf{x}+\mathbf{r}$$

Under modest assumptions one can commute the $\mathbf{x}$- and $\mathbf{x}'$-integrals. For example, this is possible if absolute integrability holds

$$\int d^dx \int d^dx'\, |G_\ell(\mathbf{x}-\mathbf{x}') f(\mathbf{x}')| < +\infty,$$

which will generally be true. In that case

$$\int d^dx\, \bar{f}_\ell(\mathbf{x}) = \int d^dx'\, f(\mathbf{x}') \left[\int d^dx\, G_\ell(\mathbf{x}-\mathbf{x}')\right].$$

However, $\int d^dx\, G_\ell(\mathbf{x}-\mathbf{x}') = \underline{1}$ by the normalization condition on G. Thus,

$$\int d^dx\, \bar{f}_\ell(\mathbf{x}) = \int d^dx'\, f(\mathbf{x}').$$

Problem 2. We shall use

$$(f*g)(x) = \int d^d r \; f(r)\, g(x-r)$$

$$= \lim_{\Delta r \to 0} \sum_{i=1}^{N} (\Delta r)^d f(r_i)\, g(x-r_i)$$

where $\sum_{i=1}^{N} (\Delta r)^d f(r_i) g(x-r_i)$ is a finite Riemann sum which approximates the integral. Now

$$\left\| \sum_{i=1}^{N} (\Delta r)^d f(r_i)\, g(\cdot - r_i) \right\|_p \leq \sum_{i=1}^{N} (\Delta r)^d \, |f(r_i)| \cdot \| g(\cdot - r_i) \|_p$$

by the triangle inequality. Since the Lp-norm itself is shift-invariant, $\| g(\cdot - r_i)\|_p = \|g\|_p$ for all i, and thus

$$\left\| \sum_{i=1}^{N} (\Delta r)^d f(r_i)\, g(\cdot - r_i) \right\|_p \leq \left[\sum_{i=1}^{N} (\Delta r)^d \, |f(r_i)| \right] \cdot \|g\|_p$$

Taking the limit $\Delta r \to 0$ and using (or assuming) that the Riemann sum converges to the integral in the Lp-norm sense,

$$\| f*g \|_p = \left\| \lim_{\Delta r \to 0} \sum_{i=1}^{N} f(r_i)\, g(\cdot - r_i) \right\|_p$$

$$\leq \lim_{\Delta r \to 0} \left\| \sum_{i=1}^{N} f(r_i)\, g(\cdot - r_i) \right\|_p$$

(cont'd)

$$\leq \lim_{\Delta r \to 0} \left[\sum_{i=1}^{N} (\Delta r)^d \, |f(r_i)| \right] \cdot \|g\|_p$$

$$= \int d^d r \, |f(r)| \cdot \|g\|_p = \|f\|_1 \cdot \|g\|_p$$

QED

<u>NOTE:</u> This is not, of course, a rigorous argument according to the standards of mathematics. For example, we assumed without any proof that

$$\left\| \lim R_N \right\|_p = \lim \| R_N \|_p$$

where R_N is the Riemann sum with N-points. Also, we should use the more general notion of the Lebesgue integral, not the Riemann integral at all. One way to derive the inequality rigorously is to derive first the special cases

$$\|f * g\|_1 \leq \|f\|_1 \cdot \|g\|_1$$

and

$$\|f * g\|_\infty \leq \|f\|_1 \cdot \|g\|_\infty$$

and then to use a general method called "Riesz-Thorin interpolation" to deduce the general case for $1 \leq p \leq +\infty$. (In fact, the general "Young inequality for convolutions" can be obtained by a second application of Riesz-Thorin interpolation.) The above "proof" should give however an intuitive understanding of the result, as a continuous generalization of the triangle inequality.

Problem 3. (a) Note that

$$E(X_1),\ E(X_1X_2),\ E(X_1X_2X_3)$$

are invariant under permutation of their arguments, by commutativity of ordinary multiplication. Furthermore,

$$C(X_1, X_2) = E(X_1X_2) - E(X_1)\cdot E(X_2)$$

is permutation invariant, since both $E(X_1X_2)$ and $E(X_1)$, $E(X_2)$ are so. The same argument then applies to

$$C(X_1, X_2, X_3) = E(X_1X_2X_3)$$
$$-\left[E(X_1)\,C(X_2,X_3) + E(X_2)\,C(X_1,X_3) + E(X_3)\,C(X_1,X_2)\right]$$
$$- E(X_1)\cdot E(X_2)\cdot E(X_3)),$$

since each of the three terms on the righthand side are now seen to be permutation invariant.

NOTE: This argument can be generalized to an inductive proof that all cumulants $C(X_1,\dots,X_n)$ are permutation-invariant for every n. However, there are easier proofs!

(b) Note that

$$C(X_1+a_1) = E(X_1+a_1)$$

$$= \int (X_1+a_1)\,dP = \int X_1\,dP + a_1 \cancel{\int dP}^{\,1} = E(X_1)+a_1$$

$$C(X_1+a_1, X_2+a_2) = E\big((X_1+a_1)(X_2+a_2)\big) - E(X_1+a_1)E(X_2+a_2)$$

$$= E(X_1X_2 + a_1X_2 + a_2X_1 + a_1a_2)$$

$$- \left[E(X_1)+a_1\right]\left[E(X_2)+a_2\right]$$

$$= \left[E(X_1X_2) + a_1E(X_2) + a_2E(X_1) + a_1a_2\right]$$

$$- \left[E(X_1)E(X_2) + a_1E(X_2) + a_2E(X_1) + a_1a_2\right]$$

$$= E(X_1X_2) - E(X_1)E(X_2) = C(X_1, X_2)$$

For $n=3$, we use

$$C(X_1+a_1, X_2+a_2, X_3+a_3) = E\big((X_1+a_1)(X_2+a_2)(X_3+a_3)\big)$$

$$- E(X_1+a_1)C(X_2,X_3) - E(X_2+a_2)C(X_1,X_3) - E(X_3+a_3)C(X_1,X_2)$$

$$- E(X_1+a_1)E(X_2+a_2)E(X_3+a_3) \qquad (*)$$

Then, we use

$$E((X_1+a_1)(X_2+a_2)(X_3+a_3))$$

$$= E(X_1X_2X_3) + a_1 E(X_2X_3) + a_2 E(X_1X_3) + a_3 E(X_1X_2)$$
$$+ a_1a_2E(X_3) + a_1a_3 E(X_2) + a_2a_3 E(X_1) + a_1a_2a_3$$

and

$$E(X_1+a_1)E(X_2+a_2)E(X_3+a_3)$$

$$= E(X_1)E(X_2)E(X_3) + a_1 E(X_1)E(X_2) + a_2 E(X_1)E(X_3) + a_3 E(X_1)E(X_2)$$
$$+ a_1a_2 E(X_3) + a_1a_3 E(X_2) + a_2a_3 E(X_1) + a_1a_2a_3$$

to get

$$E((X_1+a_1)(X_2+a_2)(X_3+a_3)) - E(X_1+a_1)E(X_2+a_2)E(X_3+a_3)$$

$$= E(X_1X_2X_3) - E(X_1)E(X_2)E(X_3)$$
$$+ a_1 C(X_1,X_2) + a_2 C(X_1,X_3) + a_3 C(X_1,X_2).$$

Finally, we obtain from (*) that

$$C(X_1+a_1, X_2+a_2, X_3+a_3) = E(X_1X_2X_3) - E(X_1)C(X_2,X_3)$$
$$- E(X_2)C(X_1,X_3) - E(X_3)C(X_1,X_2)$$
$$- E(X_1)E(X_2)E(X_3) = C(X_1,X_2,X_3).$$

QED

Problem 4, (a)

$$\tilde{\bar{f}}(x) = \int d^d r\, \tilde{G}(r)\, \bar{f}(x+r)$$

$$= \int d^d r\, \tilde{G}(r) \left[\int d^d r'\, \bar{G}(r')\, f(x+r+r') \right]$$

Now, set

$$r'' = r + r', \quad d^d r'' = d^d r'$$

and write

$$\tilde{\bar{f}}(x) = \int d^d r\, \tilde{G}(r) \int d^d r''\, \bar{G}(r''-r)\, f(x+r'')$$

$$= \int d^d r'' \left[\underbrace{\int d^d r\, \tilde{G}(r)\, \bar{G}(r''-r)}_{(\tilde{G} * \bar{G})(r'')} \right] f(x+r'')$$

by interchanging integrals and using the definition of convolution.

Thus,

$$\tilde{\bar{G}}(r'') = (\tilde{G} * \bar{G})(r'').$$

QED

(b)
$$\widetilde{\overline{\tau}}(f,g) = \widetilde{\overline{fg}} - \widetilde{\overline{f}}\,\widetilde{\overline{g}}$$
$$= \left[\widetilde{\overline{fg}} - \widetilde{\overline{f}\,\overline{g}}\right] + \left[\widetilde{\overline{f}\,\overline{g}} - \widetilde{\overline{f}}\,\widetilde{\overline{g}}\right]$$
$$= \left[\widetilde{\overline{fg} - \overline{f}\,\overline{g}}\right] + \left[\widetilde{\overline{f}\,\overline{g}} - \widetilde{\overline{f}}\,\widetilde{\overline{g}}\right]$$
$$= \widetilde{\overline{\tau}(f,g)} + \widetilde{\tau}(\overline{f},\overline{g})$$

QED

<u>Problem 5</u>. By setting $i=j$ and dividing by 2 in the evolution equation for stress, one obtains, with $k_i = \frac{1}{2}\tau_{ii}$ (no summation on i)

$$\partial_t k_i + \partial_k\left(\tfrac{1}{2}J_{iik}\right) = -\,\overline{u}_{i,k}\tau_{ki}$$
$$+\ \tau(p, S_{ii}) - \nu\,\tau(u_{i,k}, u_{i,k}) + \tau(u_i, f_i)$$

The pressure-strain term

$$\tau(p, S_{ii}) = \tau\left(p, \frac{\partial u_i}{\partial x_i}\right) \quad (\text{no summation on } i)$$

cancels when summed over i

$$\sum_i \tau(p, S_{ii}) = 0.$$

Thus, there is no net gain or loss of small-scale kinetic energy $k = \sum_i k_i$ by the pressure-strain term, which, instead, simply transfers energy between different components of the velocity. In an anisotropic flow such as a shear flow, this term is mainly responsible for the transfer of energy from the "main component" — which absorbs energy directly from the mean shear — and into the "minor components." See Tennekes & Lumley, p. 74.

Problem 6. We shall prove by induction that, for $r<1$,

$$\frac{\partial^n}{\partial r_{i_1} \cdots \partial r_{i_n}} G(\mathbf{r}) = \exp\left(\frac{-1}{1-r^2}\right) \frac{P(\mathbf{r})}{(1-r^2)^{2n}}$$

where $P(\mathbf{r})$ is a polynomial in the coordinates $r_1, \ldots, r_n$. First, for $n=1$

$$\frac{\partial}{\partial r_i} G(\mathbf{r}) = \exp\left(\frac{-1}{1-r^2}\right) \frac{-2r_i}{(1-r^2)^2}$$

by the chain rule. This gives the result with $P(\mathbf{r}) = -2r_i$.

Now assume the result for n and consider

$$\frac{\partial^{n+1}}{\partial r_{i_1}\cdots\partial r_{i_n}\partial r_{i_{n+1}}} G(\mathbf{r}) = \frac{\partial}{\partial r_{i_{n+1}}}\left[\exp\left(-\frac{1}{1-r^2}\right)\frac{P(\mathbf{r})}{(1-r^2)^{2n}}\right]$$

$$= \exp\left(\frac{-1}{1-r^2}\right)\left[\frac{-2r_{i_{n+1}}P(\mathbf{r})}{(1-r^2)^{2(n+1)}} + \frac{\frac{\partial}{\partial r_{i_{n+1}}}P(\mathbf{r})}{(1-r^2)^{2n}}\right.$$

$$\left. + P(\mathbf{r})\frac{4nr_{i_{n+1}}}{(1-r^2)^{2n+1}}\right]$$

$$= \exp\left(\frac{-1}{1-r^2}\right)\frac{\overline{P}(\mathbf{r})}{(1-r^2)^{2(n+1)}}$$

with

$$\overline{P}(\mathbf{r}) = -2r_{i_{n+1}}P(\mathbf{r}) + (1-r^2)^2\frac{\partial}{\partial r_{i_{n+1}}}P(\mathbf{r}) + 4nr_{i_{n+1}}(1-r^2)P(\mathbf{r})$$

a polynomial in the coordinates. This gives the stated result, by induction.

We now consider the limit from below

$$\lim_{r\to 1^-}\frac{\partial^n}{\partial r_{i_1}\cdots\partial r_{i_n}}G(\mathbf{r}) = \lim_{r\to 1^-}\exp\left(\frac{-1}{1-r^2}\right)\frac{P(\mathbf{r})}{(1-r^2)^{2n}}$$

One can see that the denominators go to zero

$$(1-r^2)^{2n} \longrightarrow 0$$

but the numerators go to zero even faster

$$\exp\left(\frac{-1}{1-r^2}\right) \longrightarrow 0 \quad \text{faster than } \underline{\text{any}} \text{ polynomial}$$

This may be shown by l'Hôpital's rule (since the derivatives of $(1-r^2)^{2n}$ becomes constant, but the derivatives of the numerator becomes $\exp\left(\frac{-1}{1-r^2}\right)$ times a polynomial.).
Another approach is to take logarithms and to note that

$$\ln\left[\exp\left(\frac{-1}{1-r^2}\right)\frac{P(r)}{(1-r^2)^{2n}}\right]$$

$$= \frac{-1}{1-r^2} + \ln P(r) - 2n\ln(1-r^2)$$

$$\longrightarrow -\infty \quad \text{as} \quad r \to 1^-$$

since the term $\frac{-1}{1-r^2}$ is the largest in magnitude. From all of these arguments, we see that

$$\lim_{r \to 1^-} \frac{\partial^n}{\partial r_{i_1} \cdots \partial r_{i_n}} G(\mathbf{r}) = 0$$

for all integers n. Thus, the definition

$$G(\mathbf{r}) = \begin{cases} \exp\left(-\frac{1}{1-r^2}\right) & r < 1 \\ 0 & r \geq 1 \end{cases}$$

has all nth-derivatives continuous at $r = 1$. Clearly, $G(\mathbf{r})$ is C^∞ for $r < 1$ and $r > 1$. Thus, we have proved that $G(\mathbf{r})$ is C^∞ for <u>all</u> values of r.

<u>QED</u>

Problem 7. (a) Observing as in the classnotes that

$$2\nu(\nabla\cdot\mathbf{S})_i = \nu\partial_j(\partial_i u_j + \partial_j u_i)$$
$$= \nu(0 + \partial_j^2 u_i) = \nu(\Delta\mathbf{v})_i$$

we see that the two forms of the momentum equation

$$\partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p + \nu\Delta\mathbf{v}$$

and

$$\partial_t\mathbf{v} + \nabla\cdot(\mathbf{v}\mathbf{v} + p\mathbf{I} - 2\nu\mathbf{S}) = 0$$

are equivalent for smooth solutions with the condition $\nabla\cdot\mathbf{v} = 0$. These two equations thus also imply the kinetic energy balance equation as discussed in the classnotes

$$(1) \quad \partial_t\left(\tfrac{1}{2}|\mathbf{v}|^2\right) + \nabla\cdot\left[\left(\tfrac{1}{2}|\mathbf{v}|^2 + p\right)\mathbf{v} - 2\nu\mathbf{S}\cdot\mathbf{v}\right] = -2\nu|\mathbf{S}|^2$$
$$= -\varepsilon.$$

Writing the Fourier temperature equation in the form

$$(2) \quad \partial_t(c_pT) + \nabla\cdot\left(c_pT\mathbf{v} - (\kappa/\rho)\nabla T\right) = \varepsilon,$$

we see that the sum of the last two equations gives

$$(3) \quad \partial_t\left(\tfrac{1}{2}|\mathbf{v}|^2 + c_pT\right) + \nabla\cdot\Big[\left(\tfrac{1}{2}|\mathbf{v}|^2 + c_pT + p\right)\mathbf{v}$$
$$- 2\nu\mathbf{S}\cdot\mathbf{v} - (\kappa/\rho)\nabla T\Big] = 0$$

in conservation form.

On the other hand, the conservation equation (3) for total energy per mass $\frac{1}{2}|\mathbf{v}|^2 + c_p T$ minus the equation (1) for kinetic energy per mass gives the equation (2) which is equivalent to the temperature equation

$$\partial_t T + (\mathbf{v}\cdot\nabla)T = \lambda_T \Delta T + \varepsilon/c_p .$$

Thus, the two systems of equations are entirely equivalent.

(b) We can evaluate the term including κ as

$$\nabla\cdot\left[\overline{(\kappa/\rho\,\nabla T)}_\ell\right] = -\frac{1}{\ell\rho}\int d^d r\,(\nabla G)_\ell(\mathbf{r})\ \kappa(\mathbf{x}+\mathbf{r})\nabla T(\mathbf{x}+\mathbf{r}).$$

$$= -\frac{1}{\ell\rho}\int d^d r\ 1_{\mathrm{supp}G_\ell}(\mathbf{r})\sqrt{\kappa(\mathbf{x}+\mathbf{r})}$$

$$(\nabla G)_\ell(\mathbf{r})\times\sqrt{\kappa(\mathbf{x}+\mathbf{r})}\ \nabla T(\mathbf{x}+\mathbf{r})$$

so that applying Cauchy-Schwartz inequality gives

$$\left|\nabla\cdot\left[\overline{(\kappa/\rho\,\nabla T)}_\ell(\mathbf{x})\right]\right| \leq \frac{1}{\ell\rho}\sqrt{\int_{\mathrm{supp}G_\ell} d^d r\ \kappa(\mathbf{x}+\mathbf{r})}$$

$$\times\sqrt{\int d^3 r\,|(\nabla G)_\ell(\mathbf{r})|^2\,\kappa(\mathbf{x}+\mathbf{r})\,|\nabla T(\mathbf{x}+\mathbf{r})|^2}$$

If we assume that, for fixed ℓ,

$$\int_{\mathrm{supp}\,G_\ell} d^d r \;\; \kappa(\mathbf{x}+\mathbf{r}) \longrightarrow 0$$

with

$$\int d^3x \;\; \varphi(\mathbf{x}) \;\; \kappa(\mathbf{x}) \, |\nabla T(\mathbf{x})|^2$$

remaining finite for compactly-supported, C^∞ functions φ, then we see that, for fixed ℓ,

$$|\nabla \cdot [\, (\kappa/\rho) \nabla T)_\ell(\mathbf{x}) \,]| \longrightarrow 0 .$$

In a similar fashion, the term

$$\nabla \cdot \left[\, 2\overline{(\nu \mathbf{S} \cdot \mathbf{v})_\ell} \, \right] = -\frac{1}{\ell} \int d^d r \; (\nabla G)_\ell(\mathbf{r}) \cdot \nu(\mathbf{x}+\mathbf{r}) \cdot \mathbf{S}(\mathbf{x}+\mathbf{r}) \mathbf{v}(\mathbf{x}+\mathbf{r})$$

$$= -\frac{1}{\ell} \int d^d r \;\; 1_{\mathrm{supp}\,G_\ell}(\mathbf{r}) \sqrt{\nu(\mathbf{x}+\mathbf{r})} \;\; v_i(\mathbf{x}+\mathbf{r}) \times \sqrt{\nu(\mathbf{x}+\mathbf{r})} \; S_{ij}(\mathbf{x}+\mathbf{r}) \; (\partial_j G)_\ell(\mathbf{r}) \, .$$

Applying Cauchy-Schwartz again with the above factorization of the integrand yields

$$\left|\nabla\cdot\left[2\overline{(\nu\mathbf{S}\cdot\mathbf{v})}_\ell(\mathbf{x})\right]\right| \leqslant \frac{1}{\ell}\sqrt{\int_{\mathrm{supp}G_\ell} d^d r\, \nu(\mathbf{x}+\mathbf{r})\,|\mathbf{v}(\mathbf{x}+\mathbf{r})|^2}$$

$$\times\sqrt{\int d^3 r\, |(\nabla G)_\ell(\mathbf{r})|^2\, \nu(\mathbf{x}+\mathbf{r})\,|\mathbf{S}(\mathbf{x}+\mathbf{r})|^2}$$

Thus, assuming that

$$(*)\qquad \int_{\mathrm{supp}G_\ell} d^d r\, \nu(\mathbf{x}+\mathbf{r})\,|\mathbf{v}(\mathbf{x}+\mathbf{r})|^2 \longrightarrow 0$$

and that $\int d^3x\, \varphi(\mathbf{x})\,\nu(\mathbf{x})\,|\mathbf{S}(\mathbf{x})|^2$ remains finite for smooth, compactly supported test functions, we see that

$$\left|\nabla\cdot\left[2\overline{(\nu\mathbf{S}\cdot\mathbf{v})}_\ell(\mathbf{x})\right]\right| \longrightarrow 0 .$$

Note that $(*)$ follows because

$$\int_{\mathrm{supp}G_\ell} d^d r\, \nu(\mathbf{x}+\mathbf{r})\,|\mathbf{v}(\mathbf{x}+\mathbf{r})|^2 \leqslant \left[\sup_{\mathbf{r}\in\mathrm{supp}G_\ell} \nu(\mathbf{x}+\mathbf{r})\right]\int d^d r\, |\mathbf{v}(\mathbf{x}+\mathbf{r})|^2$$

with total kinetic energy $\int d^d r\, |\mathbf{v}(\mathbf{x}+\mathbf{r})|^2$ finite and $\sup_{\mathbf{r}\in\mathrm{supp}G_\ell} \nu(\mathbf{x}+\mathbf{r}) \longrightarrow 0$.