Homework No.4, 553.793, Due March 7, 2025.

1. The spatial coarse-graining (low-pass filtering) at length-scale ℓ, which was defined in class as

$$\overline{f}_\ell(\mathbf{x}) = \int d^d r\, G_\ell(\mathbf{r}) f(\mathbf{x}+\mathbf{r}),$$

has two fundamental properties that we shall use repeatedly in this course. Prove both of these:

(1) If f is differentiable, then $\boldsymbol{\nabla}\overline{f}_\ell(\mathbf{x}) = \overline{(\boldsymbol{\nabla} f)}_\ell(\mathbf{x})$.

(2) If f is integrable, then $\int \overline{f}_\ell(\mathbf{x})\, d^d x = \int f(\mathbf{x})\, d^d x$.

2. In the definition of the convolution of two functions f and g

$$(f * g)(\mathbf{x}) = \int d\mathbf{r}\ f(\mathbf{r}) g(\mathbf{x}-\mathbf{r}),$$

approximate the integral over $\mathbf{r}$ as a Riemann sum in order to argue that its L_p-norm is bounded for $p \geq 1$ by

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p.$$

Remark: We shall use this inequality repeatedly in the course. It is a special case of the *Young inequality for convolutions*, $\|f * g\|_r \leq \|f\|_p \|g\|_q$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$.

3. For any sequence of random variables $X_1, X_2, X_3, ...$, the *cumulants* $C(X_1), C(X_1, X_2)$, $C(X_1, X_2, X_3)$ are defined iteratively in terms of the moments $E(X_1 \cdots X_k)$ by

$$E(X_1) = C(X_1)$$

$$E(X_1 X_2) = C(X_1, X_2) + C(X_1)C(X_2)$$

$$\begin{aligned} E(X_1 X_2 X_3) = C(X_1, X_2, X_3) &+ C(X_1)C(X_2, X_3) + C(X_2)C(X_1, X_3) \\ &+ C(X_3)C(X_1, X_2) + C(X_1)C(X_2)C(X_3) \end{aligned}$$

and so forth. Using the above formulas, verify that

(a) $C(X_1, X_2)$ and $C(X_1, X_2, X_3)$ do not change under permutations of their arguments, i.e. $C(X_1, X_2) = C(X_2, X_1)$, etc.

(b) Under shifts by constants a_1, a_2, a_3, $C(X_1 + a_1) = C(X_1) + a_1$, $C(X_1 + a_1, X_2 + a_2) = C(X_1, X_2)$, and $C(X_1 + a_1, X_2 + a_2, X_3 + a_3) = C(X_1, X_2, X_3)$.

Remark: We shall later develop a more powerful approach to establish such results, based upon *cumulant generating functions*.

4. This problem involves repeated filtering and a simple (but useful) formula known in LES literature as the *Germano identity*.

(a) Consider filter kernels $\widetilde{G}(\mathbf{r})$ and $\overline{G}(\mathbf{r})$, and corresponding filtering operations

$$\widetilde{f}(\mathbf{x}) = \int d\mathbf{r}\ \widetilde{G}(\mathbf{r})f(\mathbf{x}+\mathbf{r}), \quad \overline{f}(\mathbf{x}) = \int d\mathbf{r}\ \overline{G}(\mathbf{r})f(\mathbf{x}+\mathbf{r}).$$

Show that the double filtering operation $\widetilde{\overline{f}}(\mathbf{x})$ corresponds to a kernel given by the convolution of the two kernels, $\widetilde{\overline{G}} = \widetilde{G} * \overline{G}$.

(b) Define $\overline{\tau}(f,g) = \overline{fg} - \overline{f}\,\overline{g}$, associated to $\overline{G}$ and similarly for other kernels. Prove the "Germano identity":

$$\widetilde{\overline{\tau}}(f,g) = \widetilde{\overline{\tau}(f,g)} + \widetilde{\tau}(\overline{f},\overline{g}).$$

5. The pressure-strain correlation $\tau_\ell(p, S_{ij})$ drops out of the balance equation for subscale kinetic energy $k_\ell = (1/2)\mathrm{tr}\,\boldsymbol{\tau}_\ell$. However, this does not mean that it plays no role whatsoever in turbulent energy cascade. By considering separate equations for the energy in distinct velocity components $k_{\ell i} = (1/2)\tau_\ell(u_i, u_i)$ (with no summation over repeated i), try to explain the role of the pressure-strain in turbulence energetics. *Hint*: Read Tennekes & Lumley, Section 3.2 on pure shear flows, p.74.

6. Prove that the spherically-symmetric function

$$G(\mathbf{r}) = \begin{cases} \exp\left(\frac{-1}{1-r^2}\right) & r < 1 \\ 0 & r \geq 1 \end{cases}$$

is infinitely-differentiable, or C^∞. In particular, show that the function and all of its derivatives are continuous at $r = 1$.

Remark: This is a simple example of a function that is C^∞ with compact support.

7. (a) Show that the incompressible Navier-Stokes-Fourier system

$$\partial_t\mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = -\boldsymbol{\nabla}p + \nu\triangle\mathbf{v}, \quad \nabla\cdot\mathbf{v} = 0$$

$$\partial_t T + (\mathbf{v}\cdot\boldsymbol{\nabla})T = \lambda_T\triangle T + \varepsilon/c_P$$

with $\varepsilon = 2\nu|\mathbf{S}|^2$ is equivalent to the following system of conservation laws for mass, momentum, and energy:

$$\partial_t\mathbf{v} + \nabla\cdot(\mathbf{v}\mathbf{v} + p\mathbf{I} - 2\nu\mathbf{S}) = \mathbf{0}, \quad \nabla\cdot\mathbf{v} = 0$$

$$\partial_t\left(\frac{1}{2}|\mathbf{v}|^2 + c_P T\right) + \nabla\cdot\left[\left(\frac{1}{2}|\mathbf{v}|^2 + c_P T + p\right)\mathbf{v} - 2\nu\mathbf{S}\cdot\mathbf{v} - (\kappa/\rho)\nabla T\right] = 0$$

where ρ is the constant mass-density and $\kappa = \rho c_P \lambda_T$ is the thermal conductivity.

(b) If the energy conservation equation is coarse-grained at length-scale ℓ, giving

$$\partial_t \overline{\left(\frac{1}{2}|\mathbf{v}|^2 + c_P T\right)}_\ell + \nabla \cdot \overline{\left[\left(\frac{1}{2}|\mathbf{v}|^2 + c_P T + p\right)\mathbf{v} - 2\nu\mathbf{S}\cdot\mathbf{v} - (\kappa/\rho)\nabla T\right]}_\ell = 0,$$

then show that pointwise

$$\boldsymbol{\nabla}\boldsymbol{\cdot}\overline{(2\nu\mathbf{S}\boldsymbol{\cdot}\mathbf{v})}_\ell,\ \ \boldsymbol{\nabla}\boldsymbol{\cdot}\overline{((\kappa/\rho)\nabla T)}_\ell \to 0$$

in the joint limit $\nu,\ \kappa \to 0$. In this conservation form, we can take transport coefficients to be space-time dependent functions $\nu(\mathbf{x})$, $\kappa(\mathbf{x})$ and we assume that

$$\sup_{\mathbf{x}\in K} \nu(\mathbf{x}),\ \ \sup_{\mathbf{x}\in K} \kappa(\mathbf{x}) \to 0$$

for any compact (closed, bounded) sets K. We also assume that viscous dissipation rates and thermal dissipation rates smeared with C^∞, compactly-supported test functions φ remain finite

$$\int d^dx\ \varphi(\mathbf{x})\ 2\nu(\mathbf{x})|\mathbf{S}(\mathbf{x})|^2 < \infty, \qquad \int d^dx\ \varphi(\mathbf{x})\ \kappa(\mathbf{x})|\boldsymbol{\nabla}T(\mathbf{x})|^2 < \infty,$$

in the limit. *Hint:* Use the Cauchy-Schwartz inequality.