Homework # 3 – Solutions

Problem 1. Assume that p solves

$$-\Delta p(x,t) = \partial_i \partial_j (u_i(x,t) u_j(x,t)) .$$

Substituting

$$u(x,t) = \lambda^{-h} u_\lambda (\lambda x, \lambda^{1-h} t)$$

gives

$$-\Delta p(x,t) = \lambda^{2-2h} \tilde{\partial}_i \tilde{\partial}_j (u_{\lambda i}(\tilde{x},\tilde{t}) u_{\lambda j}(\tilde{x},\tilde{t}))$$

with $\tilde{x} = \lambda x$, $\tilde{t} = \lambda^{1-h} t$. Making the same change of variables on the LHS gives

$$-\lambda^2 \tilde{\Delta} \cdot p(\lambda^{-1}\tilde{x}, \lambda^{h-1}\tilde{t}) = \lambda^{2-2h} \tilde{\partial}_i \tilde{\partial}_j (u_{\lambda i}(\tilde{x},\tilde{t}) u_{\lambda j}(\tilde{x},\tilde{t}))$$

or

$$-\tilde{\Delta} p_\lambda(\tilde{x},\tilde{t}) = \tilde{\partial}_i \tilde{\partial}_j (u_{\lambda i}(\tilde{x},\tilde{t}) u_{\lambda j}(\tilde{x},\tilde{t}))$$

with

$$p_\lambda(\tilde{x},\tilde{t}) \equiv \lambda^{2h} p(\lambda^{-1}\tilde{x}, \lambda^{h-1}\tilde{t})$$

the solution of the Poisson equation corresponding to velocity $u_\lambda(x,t)$. Taking the gradient $\tilde{\nabla}$ (with respect to $\tilde{x}$) and using the chain rule gives

$$\tilde{\nabla} p_\lambda(\tilde{x},\tilde{t}) = \lambda^{2h-1} (\nabla p)(\lambda^{-1}\tilde{x}, \lambda^{h-1}\tilde{t})$$

QED

Problem 2 (a) The fluid velocity $\mathbf{v}'(\mathbf{x},t)$ in the frame moving with velocity $\mathbf{U} = U\hat{\mathbf{e}}_x$ relative to the wind-tunnel frame of reference is given by

$$\mathbf{v}'(\mathbf{x},t) = \mathbf{v}(\mathbf{x}+\mathbf{U}t, t) - \mathbf{U}$$

in terms of the fluid velocity $\mathbf{v}(\mathbf{x},t)$ in the lab frame. Retaining only x-components, for simplicity of notations, we can write this as

$$u'(x,t) = u(x+Ut,t) - U.$$

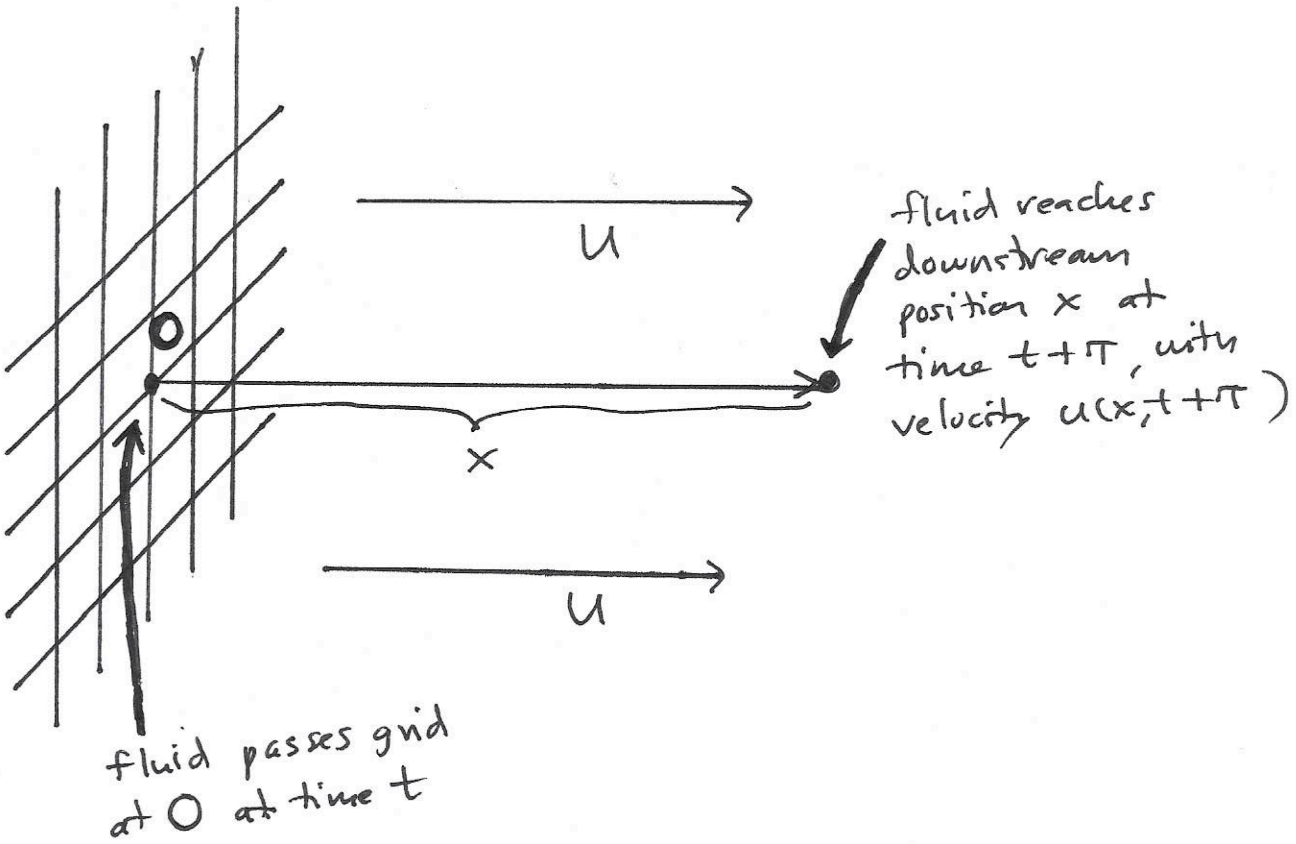
(b) For $a(x,t)$ which is any local function such as $\partial_x u(x,t)$ or $u(x+\ell,t)-u(x,t)$ in which the contribution of $-U$ vanishes, the relation between the two frames is given by

$$a'(x,t) = a(x+Ut,t).$$

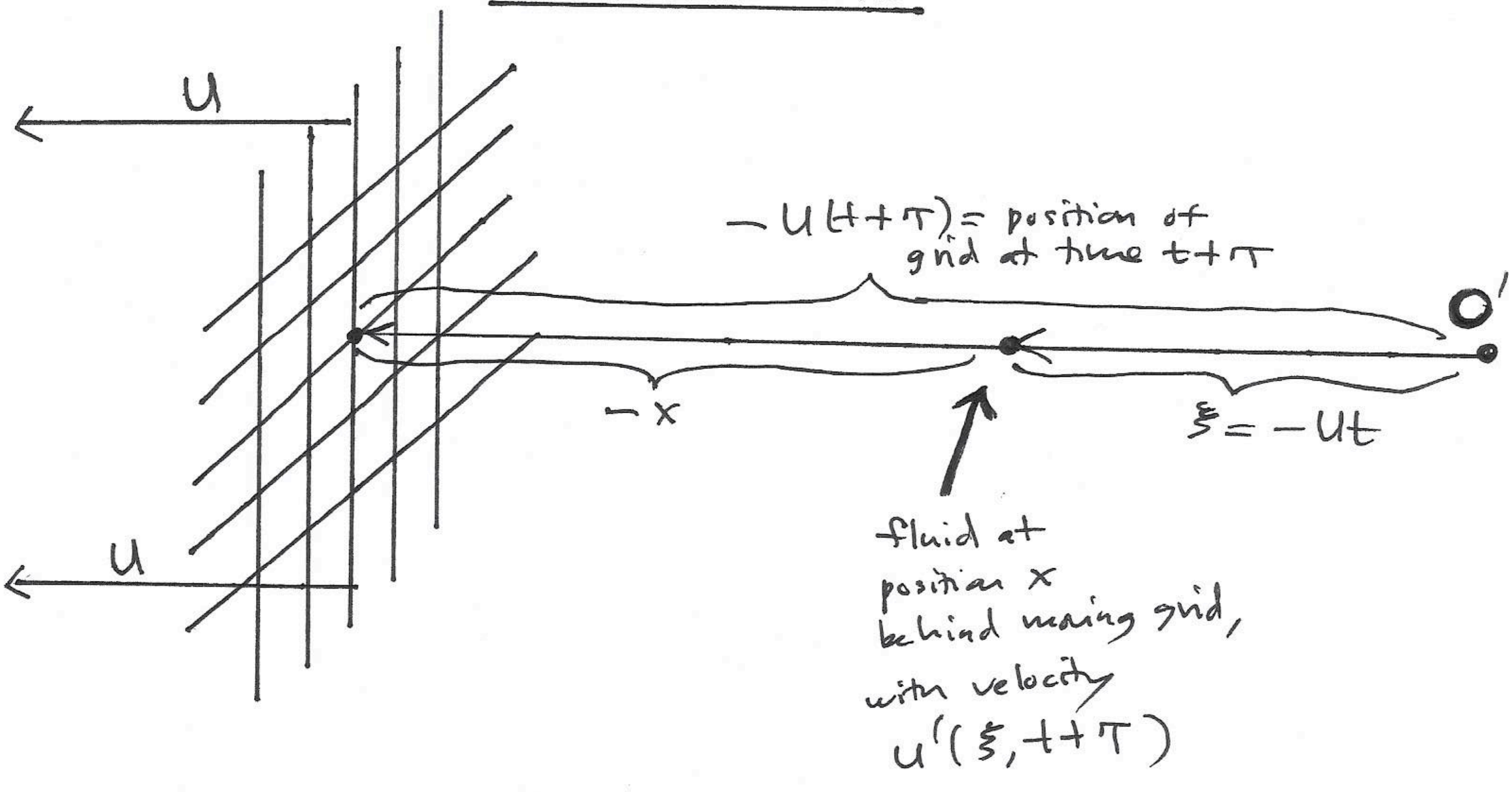
Thus, defining $\tau := x/U$, $\xi := -Ut$

$$\begin{aligned} a(x,t+\tau) &= a(U\tau, t+\tau) \\ &= a(-Ut + U(t+\tau), t+\tau) \\ &= a'(-Ut, t+\tau) \\ &= a'(\xi, t+\tau) \end{aligned}$$

wind-tunnel frame



fluid rest-frame



(c) Note that

$$\frac{d}{dt} a'(-Ut, t+\tau)$$

$$= \left(-U\partial_x + \partial_t\right) a'(-Ut, t+\tau)$$

$$\doteq -U\partial_x a'(-Ut, t+\tau)$$

when $U \gg |u'|$. Thus, to leading order, we can neglect the internal time-dependence from the nonlinear evolution of the fluid and consider only the time-dependence of the rapid sweeping of the grid, so that

$$a'(-Ut, t+\tau) \doteq a'(-Ut, \tau)$$

Exploiting this approximation, or

$$a'(\xi, t+\tau) = a'(\xi, \tau),$$

one finds that

$$\langle a(x) \rangle = \frac{1}{T} \int_0^T dt \; a(x, t+\tau)$$

(cont'd)

$$= \frac{1}{T}\int_0^T dt\ a'(-Ut, t+\tau)$$

$$\doteq \frac{1}{T}\int_0^T dt\ a'(-Ut, \tau) \quad \text{by Taylor hypothesis}$$

$$= \frac{1}{L}\int_{-L}^{0} d\xi\ a'(\xi, \tau) \qquad \text{w/ } L = UT,$$

by the change of integration variable from t to $\xi = -Ut$. Note that the space variable x has been exchanged for a time variable $\tau = x/U$ and the time variable t has been exchanged for a space variable $\xi = -Ut$.

(d) In the Navier-Stokes equation in the fluid frame

$$\partial_\tau \mathbf{v}' + (\mathbf{v}'\cdot\nabla_\xi)\mathbf{v}' = -\nabla_\xi p' + \nu\Delta_\xi \mathbf{v}'$$

introduce the dimensionless variables

$$\hat{\xi} = \xi/M,\ \hat{\mathbf{v}}' = \mathbf{v}'/U,\ \hat{\tau} = U\tau/M,\ \hat{p}' = p'/U^2$$

which yields

$$\partial_{\hat{\tau}}\hat{\mathbf{v}}' + (\hat{\mathbf{v}}'\cdot\hat{\nabla}_\xi)\hat{\mathbf{v}}' = -\hat{\nabla}_\xi \hat{p}' + \frac{1}{Re_M}\hat{\Delta}_\xi \hat{\mathbf{v}}'$$

with

$$Re_M = \frac{UM}{\nu}.$$

The solution of this equation

$$\hat{\mathbf{v}}' = \hat{\mathbf{v}}'(\hat{\boldsymbol{\xi}}, \hat{T}, Re_M)$$

is a unique function of the dimensionless variables $\hat{\boldsymbol{\xi}}, \hat{T}, Re_M$ (and geometric properties of the grid).

(e) Note that with $\hat{S}'_{ij}(\hat{\xi}, \hat{T}) = \hat{S}_{ij}(\hat{\xi}, \hat{T}; Re_M)$

$$\begin{aligned}
\varepsilon'(\hat{\boldsymbol{\xi}}, \hat{T}) &= 2\nu\left(S'(\hat{\boldsymbol{\xi}}, \hat{T})\right)^2 \\
&= 2\nu\left(\frac{U}{M}\hat{S}'(\hat{\boldsymbol{\xi}}, \hat{T})\right)^2 \\
&= 2\nu\frac{U^2}{M^2}\left(\hat{S}'(\hat{\xi}, \hat{T})\right)^2 \\
&= \frac{2}{Re_M}\frac{U^3}{M}\left(\hat{S}'(\hat{\xi}, \hat{T})\right)^2 \quad \text{w/ } Re_M = \frac{UM}{\nu}
\end{aligned}$$

so that

$$\varepsilon'(\hat{\boldsymbol{\xi}}, \hat{T})/U^3/M = \frac{2}{Re_M}\left(\hat{S}'(\hat{\boldsymbol{\xi}}, \hat{T})\right)^2$$

and

$$D := \frac{\langle \varepsilon'(\hat{T})\rangle}{U^3/M} = \frac{2}{Re_M}\frac{1}{\hat{L}}\int_{\hat{L}}^{0} d\hat{\xi}\left(\hat{S}'(\hat{\xi}, \hat{T})\right)^2 \quad \text{w/ } \hat{L} = \frac{L}{M}$$

for $\hat{L} \gg 1$ becomes a function $D(\hat{T}, Re_M)$ of $\hat{T}, Re_M$ only, and $D(\hat{T})$ of $\hat{T}$ only for $Re_M \gg 1$, assuming $Re_M \to \infty$ limit exists.

(f) Since $\tau = x/U$, one gets from the general result in part (c) that $U\partial_x = \partial_\tau$ and thus

$$\langle |u'(x)|^2 \rangle \doteq \langle |u'(\tau)|^2 \rangle = \frac{1}{L}\int_{-L}^{0} d\xi\, |u'(\xi,\tau)|^2$$

$\Longrightarrow$

$$U\partial_x \langle |u'(x)|^2 \rangle \doteq \langle \partial_\tau |u'(\tau)|^2 \rangle$$

$$= \frac{1}{L}\int_{-L}^{0} d\xi\ \partial_\tau |u'(\xi,\tau)|^2$$

If one assumes that x is sufficiently far from the grid that the statistics are isotropic in the fluid frame, then

$$\frac{3}{2}\langle |u'(\tau)|^2 \rangle \doteq \frac{1}{2}\langle |\mathbf{u}'(\tau)|^2 \rangle .$$

If furthermore the statistics are homogeneous so that $\langle \nabla \cdot \mathbf{J}' \rangle = 0$ for $\mathbf{J}'$ the spatial energy flux, then energy balance gives

$$\partial_\tau \frac{1}{2}\langle |\mathbf{u}'(\tau)|^2 \rangle \doteq -\langle \varepsilon'(\tau) \rangle .$$

Putting all of these results together

$$-\frac{3}{2}U\partial_x \langle |u'(x)|^2 \rangle \doteq \langle \varepsilon'(\tau) \rangle$$

Problem 3. (a) In infinite space, the solution of

$$-\Delta D(\mathbf{r}) = \delta(\mathbf{r})$$

is

$$D(\mathbf{r}) = \frac{1}{4\pi r}.$$

Then

$$\mathbf{K}(\mathbf{r}) = \nabla D(\mathbf{r}) = \frac{-\mathbf{r}}{4\pi r^3}$$

so that

$$\mathbf{v}(\mathbf{x}) = \int \mathbf{K}(\mathbf{x}-\mathbf{x}') \times \boldsymbol{\omega}(\mathbf{x}')\, d^3x'$$

$$= \frac{1}{4\pi} \int \frac{\boldsymbol{\omega}(\mathbf{x}') \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\, d^3x'.$$

(b) Substituting $\boldsymbol{\omega}_C(\mathbf{x}) = \Gamma \oint_C d\mathbf{r}\, \delta^3(\mathbf{x}-\mathbf{r})$ into the Biot-Savart formula gives

$$\mathbf{v}_C(\mathbf{x}) = \frac{\Gamma}{4\pi} \oint_C \frac{d\mathbf{r} \times (\mathbf{x}-\mathbf{r})}{|\mathbf{x}-\mathbf{r}|^3}.$$

Integrating over $\mathbf{x}$

$$\boldsymbol{\omega}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) = \sum_{i,j=1}^{n} \boldsymbol{\omega}_{C_i}(\mathbf{x}) \cdot \mathbf{v}_{C_j}(\mathbf{x})$$

then gives

$$\int d^3x \; \boldsymbol{\omega}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x})$$

$$= \sum_{i,j=1}^{n} \Gamma_i \Gamma_j \frac{1}{4\pi} \oint_{C_i} d\mathbf{r}_i \cdot \left[\oint_{C_j} \frac{d\mathbf{r}_j \times (\mathbf{r}_i - \mathbf{r}_j)}{|\mathbf{r}_i - \mathbf{r}_j|^3} \right]$$

Using the well-known relation for the triple product

$$d\mathbf{r}_i \cdot d\mathbf{r}_j \times (\mathbf{r}_i - \mathbf{r}_j) = d\mathbf{r}_i \times d\mathbf{r}_j \cdot (\mathbf{r}_i - \mathbf{r}_j)$$

then yields

$$\int d^3x \; \boldsymbol{\omega}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) = \sum_{i,j=1}^{n} \Gamma_i \Gamma_j \left[\frac{1}{4\pi} \oint_{C_i} \oint_{C_j} (d\mathbf{r}_i \times d\mathbf{r}_j) \cdot \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3} \right]$$

$$= \sum_{i,j=1}^{n} \Gamma_i \Gamma_j \, \ell_{ij} \qquad \checkmark$$

<u>Problem 4.</u> It was shown in class that

$$\frac{d}{dt} \int_{C(t)} \mathbf{u}(t) \cdot d\mathbf{x} = \int_{C(t)} \left[D_t \mathbf{u}(t) \cdot d\mathbf{x} + d\left(\tfrac{1}{2} |\mathbf{u}|^2 \right) \right].$$

Thus, for a <u>closed</u> loop,

$$\frac{d}{dt} \oint_{C(t)} \mathbf{u}(t) \cdot d\mathbf{x} = \oint_{C(t)} D_t \mathbf{u}(t) \cdot d\mathbf{x}$$

for <u>every</u> initial closed loop C, advected by $\mathbf{u}$ to $C(t)$ at time t.

Let us suppose, then, that

$$\frac{d}{dt}\oint_{C(t)} u(t)\cdot dx = \nu \oint_{C(t)} \Delta u(t)\cdot dx$$

for every initial loop C. By the previous result

$$\oint_{C(t)} D_t u(t)\cdot dx = \nu \oint_{C(t)} \Delta u(t)\cdot dx$$

so that, subtracting the LHS from both sides,

$$\oint_{C(t)} \left[D_t \cancel{u(t)} - \nu \Delta u(t) \right]\cdot dx = 0 . \qquad (*)$$

Now, choose <u>any</u> loop Γ and, starting at time t, evolve it backward under u to time 0. Call this loop $\Gamma(-t)$. Setting $C = \Gamma(-t)$, one can easily see that for this choice, $C(t) = \Gamma$. Applying $(*)$ with this loop we get

$$\oint_{\Gamma} \left[D_t u(t) - \nu \Delta u(t) \right]\cdot dx = 0 .$$

But here Γ is <u>arbitrary</u>. Thus, there must exist some "potential" $\varphi(t)$ such that

$$D_t u(t) - \nu \Delta u(t) = \nabla \varphi(t)$$

[NOTE: This is only strictly true if the flow domain is simply connected. Otherwise, there are counterexamples.]

Clearly we may identify the "potential" φ with the negative of the pressure, $\varphi = -p$, and write

$$D_t \mathbf{u}(t) = -\nabla p(t) + \nu \Delta \mathbf{u}(t),$$

or

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \nu \Delta \mathbf{u},$$

by writing out the material derivative. Since we have assumed that the velocity field $\mathbf{u}$ is solenoidal, or $\nabla \cdot \mathbf{u} = 0$, the pressure is then uniquely determined by the Poisson equation

$$-\Delta p = \partial_i \partial_j (u_i u_j).$$

Thus, we have recovered exactly the incompressible Navier–Stokes equation.

QED