Homework No.3, 553.793, Due March 4, 2022.

1. If $u_\lambda(x,t) = \lambda^h u(\lambda^{-1}x, \lambda^{h-1}t)$ is the velocity field of a rescaled Euler solution, then show that $p_\lambda(x,t) = \lambda^{2h} p(\lambda^{-1}x, \lambda^{h-1}t)$ and $\nabla p_\lambda(x,t) = \lambda^{2h-1} \nabla p(\lambda^{-1}x, \lambda^{h-1}t)$ are the corresponding pressure and pressure-gradient from the Poisson equation.

2. This problem applies some basic properties of the incompressible Navier-Stokes equation to the analysis of decaying turbulence behind a wire-mesh grid in a wind tunnel. We let $U$ denote the inflow velocity of the fluid, taken to be in the $x$-direction, and $M$ the mesh-length of the grid.

(a) If $v(x,t)$ is the fluid velocity field in the wind-tunnel frame of reference, then write down a formula for the velocity field $v'(x,t)$ in the frame moving with the inflow velocity $U$. For simplicity, assume symmetry in the $yz$-plane parallel to the grid and just write the formula in terms of the $x$-components, $u(x,t)$ and $u'(x,t)$.

Remark: In the new reference frame, the fluid is at rest and the grid moves through the fluid in the negative $x$-direction. This situation is sometimes created in the laboratory, by so-called “towed grid” experiments.

(b) Consider any quantity $a(x,t)$ which is a local function of the velocity-field increments and gradients (e.g. the energy dissipation rate $\varepsilon(x,t) = 2\nu S^2(x,t)$), at distance $x$ downstream from the grid and at time $t$, in the lab frame. Let $\tau = x/U$ denote the time it takes for the fluid to flow past the grid at time $t$ to position $x$ downstream at time $t + \tau$. Use the result of part (a) to show that $a$ in the lab frame is related to $a'$ in the fluid frame by

$$a(x, t + \tau) = a'(-Ut, t + \tau) = a'(\xi, t + \tau)$$

where $\xi = -Ut$ is the distance the grid has travelled over time $t$ in the fluid frame.

(c) According to the Taylor frozen-turbulence hypothesis

$$a'(-Ut, t + \tau) \approx a'(-Ut, \tau)$$

when the turbulence intensity is low and $U \gg |u'|$. The intuition here is that the rate of change due to the rapid grid motion $\xi = -Ut$ is much larger than the rate due to the nonlinear evolution from $(v' \cdot \nabla)v'$. This idea was proposed by G. I. Taylor in 1935 as a simple approximation to permit study of time-evolution in grid-turbulence. Use the hypothesis to show that

$$\langle a(x) \rangle := \frac{1}{T} \int_0^T dt \ a(x, t + \tau) \approx \frac{1}{L} \int_{-L}^0 d\xi \ a'(\xi, \tau) \ := \langle a'(\tau) \rangle$$

with $L = UT$. Thus, a time-average at downstream position $x$ in the lab frame corresponds to a space-average at time $\tau$ in the fluid frame.
(d) Use the principle of hydrodynamic similarity to argue that the non-dimensional velocity \( \hat{u}' = u'(\xi, \tau)/U \) in the fluid frame is a unique function
\[
\hat{u}' = \hat{u}'(\hat{\xi}, \hat{\tau}, \hat{R_e}_M)
\]
of the dimensionless variables \( \hat{\xi} = \xi/M, \hat{\tau} = U\tau/M, \) and mesh Reynolds number \( \hat{R_e}_M = U M/\nu, \) in addition to scale-invariant geometric properties of the grid, such as the angles made by the wires where they meet and length ratios such as \( d/M, \) with \( d \) the diameter of the individual wires.

(e) Use the results of part (d) to show that the non-dimensionalized mean dissipation rate in the fluid frame
\[
D = \frac{\langle \varepsilon'(\tau) \rangle}{U^3/M}
\]
is a unique function \( D = D(\hat{\tau}, \hat{R_e}_M) \) of the variables \( \hat{\tau}, \hat{R_e}_M. \) Assume that there is no dependence upon the length \( L \) of the space-interval averaged over in part (c), because \( L \) is so large that the space-average is converged. Experiments indicate that \( D \) becomes independent of \( \hat{R_e}_M \) when \( \hat{R_e}_M \gg 1, \) but for some grid configurations it appears that \( D = D(\hat{\tau}) \) remains dependent upon \( \hat{\tau} \) even when \( \hat{R_e}_M \gg 1. \) See Vassilicos (2015).

3. (a) Show that in infinite three-dimensional space, the Biot-Savart formula becomes
\[
v(x) = \frac{1}{4\pi} \int d^3x' \frac{\omega(x') \times (x - x')}{|x - x'|^3}.
\]
(b) Suppose that \( C \) is a closed, singular vortex line with circulation \( \Gamma \) around any part of the line, whose vorticity field is
\[
\omega_C(x) = \Gamma \oint_C dr \delta^3(x - r).
\]
Now consider an assemblage of \( n \) such non-intersecting vortex loops \( C_1, ..., C_n \) and their summed fields for vorticity \( \omega(x) \) and velocity \( v(x) \). Show that their helicity is
\[
\int d^3x \omega(x) \cdot v(x) = \sum_{i,j=1}^n \ell_{ij} \Gamma_i \Gamma_j
\]
in terms of the Gauss linking number of loops \( i \) and \( j \):
\[
\ell_{ij} = \frac{1}{4\pi} \oint_{C_i} \oint_{C_j} \frac{r_i - r_j}{|r_i - r_j|^3} \cdot (dr_i \times dr_j),
\]
an integer. Remark: This formula is singular and ill-defined for the self-linking number when \( i = j. \) To make sense of it, one must consider the linkage of a pair \( C_i, C'_i \) where \( C'_i \) is a slight displacement of \( C_i. \) This yields a well-defined, unique result independent of exactly how \( C'_i \) is displaced from \( C_i. \)
4. Show that Kelvin’s circulation theorem is equivalent to the Navier-Stokes equation. That is, show that if a solenoidal velocity field ($\nabla \cdot \mathbf{u} = 0$) satisfies

$$\frac{d}{dt} \oint_{C(t)} \mathbf{u}(t) \cdot d\mathbf{x} = \nu \oint_{C(t)} \Delta \mathbf{u}(t) \cdot d\mathbf{x}$$

for every initial closed loop $C = C(t_0)$ advected by $\mathbf{u}$ to $C(t)$ at time $t$, then $\mathbf{u}(\mathbf{x}, t)$ must satisfy the Navier-Stokes equation. *Hint:* Use the result proved in class that

$$\frac{d}{dt} \oint_{C(t)} \mathbf{u}(t) \cdot d\mathbf{x} = \oint_{C(t)} D_t \mathbf{u}(t) \cdot d\mathbf{x},$$

where $D_t$ is the Lagrangian derivative and the fact that a vector field $\mathbf{f}(\mathbf{x}, t)$ is a gradient of a potential if and only if $\oint_C \mathbf{f} \cdot d\mathbf{x} = 0$ for all closed loops $C$.

Remark: This fact was already pointed out by Lord Kelvin (William Thomson) in his original paper deriving the theorem. This remarkable paper, still well worth reading, can be found on-line:

[http://empslocal.ex.ac.uk/people/staff/gv219/classics.d/Kelvin1869.pdf](http://empslocal.ex.ac.uk/people/staff/gv219/classics.d/Kelvin1869.pdf)