Homework No.3, 553.793, Due February 21, 2025.

1. If $\mathbf{u}_\lambda(\mathbf{x},t) = \lambda^h \mathbf{u}(\lambda^{-1}\mathbf{x}, \lambda^{h-1}t)$ is the velocity field of a rescaled Euler solution, then show that $p_\lambda(\mathbf{x},t) = \lambda^{2h} p(\lambda^{-1}\mathbf{x}, \lambda^{h-1}t)$ and $\boldsymbol{\nabla} p_\lambda(\mathbf{x},t) = \lambda^{2h-1}\boldsymbol{\nabla} p(\lambda^{-1}\mathbf{x}, \lambda^{h-1}t)$ are the corresponding pressure and pressure-gradient from the Poisson equation.

2. This problem applies some basic properties of the incompressible Navier-Stokes equation to the analysis of decaying turbulence behind a wire-mesh grid in a wind tunnel. We let U denote the inflow velocity of the fluid, taken to be in the x-direction, and M the mesh-length of the grid.

(a) If $\mathbf{v}(\mathbf{x},t)$ is the fluid velocity field in the wind-tunnel frame of reference, then write down a formula for the velocity field $\mathbf{v}'(\mathbf{x},t)$ in the frame moving with the inflow velocity U. For simplicity, assume symmetry in the yz-plane parallel to the grid and just write the formula in terms of the x-components, $u(x,t)$ and $u'(x,t)$.

Remark: In the new reference frame, the fluid is at rest and the grid moves through the fluid in the negative x-direction. This situation is sometimes created in the laboratory, by so-called "towed grid" experiments.

(b) Consider any quantity $a(x,t)$ which is a local function of the velocity-field increments and gradients (e.g. the energy dissipation rate $\varepsilon(x,t) = 2\nu S^2(x,t)$), at distance x downstream from the grid and at time t, in the lab frame. Let $\tau = x/U$ denote the time it takes for the fluid to flow past the grid at time t to position x downstream at time $t+\tau$. Use the result of part (a) to show that a in the lab frame is related to a' in the fluid frame by

$$a(x, t+\tau) = a'(-Ut, t+\tau) = a'(\xi, t+\tau)$$

where $\xi = -Ut$ is the distance the grid has travelled over time t in the fluid frame.

(c) According to the *Taylor frozen-turbulence hypothesis*

$$a'(-Ut, t+\tau) \doteq a'(-Ut, \tau)$$

when the turbulence intensity is low and $U \gg |u'|$. The intuition here is that the rate of change due to the rapid grid motion $\xi = -Ut$ is much larger than the rate due to the nonlinear evolution from $(\mathbf{v}'\boldsymbol{\cdot}\boldsymbol{\nabla})\mathbf{v}'$. This idea was proposed by G. I. Taylor in 1935 as a simple approximation to permit study of time-evolution in grid-turbulence. Use the hypothesis to show that

$$\langle a(x)\rangle := \frac{1}{T}\int_0^T dt\ a(x, t+\tau) \doteq \frac{1}{L}\int_{-L}^0 d\xi\ a'(\xi,\tau) := \langle a'(\tau)\rangle$$

with $L = UT$. Thus, a time-average at downstream position x in the lab frame corresponds to a space-average at time τ in the fluid frame.

(d) Use the principle of hydrodynamic similarity to argue that the non-dimensional velocity $\widehat{u}' = u'(\xi,\tau)/U$ in the fluid frame is a unique function $\widehat{u}' = \widehat{u}'(\widehat{\xi},\widehat{\tau},Re_M)$ of the dimensionless variables $\widehat{\xi} = \xi/M$, $\widehat{\tau} = U\tau/M$, and mesh Reynolds number $Re_M = UM/\nu$, in addition to scale-invariant geometric properties of the grid, such as the angles made by the wires where they meet and length ratios such as d/M, with d the diameter of the individual wires.

(e) Use the results of part (d) to show that the non-dimensionalized mean dissipation rate in the fluid frame

$$D = \frac{\langle \varepsilon'(\tau)\rangle}{U^3/M}$$

is a unique function $D = D(\widehat{\tau}, Re_M)$ of the variables $\widehat{\tau}$, Re_M. Assume that there is no dependence upon the length L of the space-interval averaged over in part (c), because L is so large that the space-average is converged.

(f) Use the result in part (c) to argue that

$$U\partial_x\langle |u'(x)|^2\rangle \doteq \partial_\tau\langle |u'(\tau)|^2\rangle = \frac{1}{L}\int_{-L} d\xi\, \partial_\tau |u'(\xi,\tau)|^2.$$

If one assumes that x is sufficiently far from the grid that the statistics of $\mathbf{u}'$ in the fluid frame are locally homogeneous and isotropic, then use the standard energy balance relation in that frame

$$\partial_\tau \frac{1}{2}|\mathbf{u}'(\boldsymbol{\xi},\tau)|^2 + \boldsymbol{\nabla}\boldsymbol{\cdot}\left[\left(\frac{1}{2}|\mathbf{u}'(\boldsymbol{\xi},\tau)|^2 + p'(\boldsymbol{\xi},\tau)\right)\mathbf{u}'(\boldsymbol{\xi},\tau) + 2\nu \mathbf{S}'(\boldsymbol{\xi},\tau)\boldsymbol{\cdot}\mathbf{u}'(\boldsymbol{\xi},\tau)\right] = -\varepsilon'(\boldsymbol{\xi},\tau)$$

to argue furthermore that

$$-\frac{3}{2}U\partial_x\langle |u'(x)|^2\rangle \doteq \langle \varepsilon'(\tau)\rangle.$$

This relationship is the entire basis of experimental studies of energy dissipation in grid turbulence by using hot-wire anemometry to measure u'.

3. (a) Show that in infinite three-dimensional space, the Biot-Savart formula becomes

$$\mathbf{v}(\mathbf{x}) = \frac{1}{4\pi}\int d^3x'\, \frac{\boldsymbol{\omega}(\mathbf{x}')\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}.$$

(b) Suppose that C is a closed, singular vortex line with circulation Γ around any part of the line, whose vorticity field is

$$\boldsymbol{\omega}_C(\mathbf{x}) = \Gamma \oint_C d\mathbf{r}\ \delta^3(\mathbf{x}-\mathbf{r}).$$

Now consider an assemblage of n such non-intersecting vortex loops $C_1, ..., C_n$ and their summed fields for vorticity $\boldsymbol{\omega}(\mathbf{x})$ and velocity $\mathbf{v}(\mathbf{x})$. Show that their helicity is

$$\int d^3x \, \boldsymbol{\omega}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) = \sum_{i,j=1}^{n} \ell_{ij} \Gamma_i \Gamma_j$$

in terms of the *Gauss linking number* of loops i and j:

$$\ell_{ij} = \frac{1}{4\pi} \int_{C_i} \int_{C_j} \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3} \cdot (d\mathbf{r}_i \times d\mathbf{r}_j),$$

an integer. *Remark:* This formula is singular and ill-defined for the *self-linking number* when $i = j$. To make sense of it, one must consider the linkage of a pair C_i, C_i' where C_i' is a slight displacement of C_i. This yields a well-defined, unique result independent of exactly how C_i' is displaced from C_i.

4. Show that Kelvin's circulation theorem is equivalent to the Navier-Stokes equation. That is, show that if a solenoidal velocity field ($\boldsymbol{\nabla} \cdot \mathbf{u} = 0$) satisfies

$$\frac{d}{dt} \oint_{C(t)} \mathbf{u}(t) \cdot d\mathbf{x} = \nu \oint_{C(t)} \triangle \mathbf{u}(t) \cdot d\mathbf{x}$$

for every initial closed loop $C = C(t_0)$ advected by $\mathbf{u}$ to $C(t)$ at time t, then $\mathbf{u}(\mathbf{x}, t)$ must satisfy the Navier-Stokes equation. *Hint:* Use the result proved in class that

$$\frac{d}{dt} \oint_{C(t)} \mathbf{u}(t) \cdot d\mathbf{x} = \oint_{C(t)} D_t \mathbf{u}(t) \cdot d\mathbf{x},$$

where D_t is the Lagrangian derivative and the fact that a vector field $\mathbf{f}(\mathbf{x}, t)$ is a gradient of a potential if and only if $\oint_C \mathbf{f} \cdot d\mathbf{x} = 0$ for all closed loops C.

Remark: This fact was already pointed out by Lord Kelvin (William Thomson) in his original paper deriving the theorem. This remarkable paper, still well worth reading, can be found on-line:

`http://empslocal.ex.ac.uk/people/staff/gv219/classics.d/Kelvin1869.pdf`