Homework No.2, 553.793, Due February 11, 2022.

1. This problem studies the local conservation laws of classical molecular fluids.

(a) Derive the local conservation of momentum

$$\partial_t \hat{\mathbf{g}}(\mathbf{x},t) + \boldsymbol{\nabla}\boldsymbol{\cdot}\hat{\mathbf{T}}(\mathbf{x},t) = \mathbf{0},$$

where $\mathbf{T}$ is the stress-tensor given in class and $(\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{T})_i = \partial_j T_{ij}$. *Hint:* Use Newton's third law, and also use the fundamental theorem of calculus to write

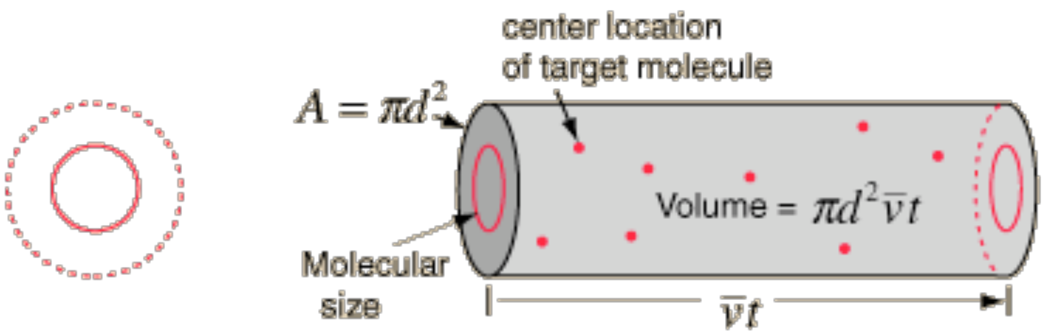
$$(\mathbf{r}_{nm}\boldsymbol{\cdot}\boldsymbol{\nabla}) \int_0^1 ds\ \delta^3(\mathbf{x}-\mathbf{r}_n + s\mathbf{r}_{nm}) = \delta^3(\mathbf{x}-\mathbf{r}_m) - \delta^3(\mathbf{x}-\mathbf{r}_n).$$

(b) Use a similar argument to derive the local conservation of energy

$$\partial_t \hat{e}(\mathbf{x},t) + \boldsymbol{\nabla}\boldsymbol{\cdot}\hat{\mathbf{s}}(\mathbf{x},t) = 0$$

where $\hat{\mathbf{s}}$ is the energy current given in class.

2. The concept of the *mean free path length* is a very important one in the molecular theory of fluids, which we explore in this problem.



(a) The above figure illustrates the notion of the "collision cylinder" swept out in time t by a spherical particle of diameter d moving at speed $\bar{v}$ in a gas of similar particles at rest. The points in the figure represent the centers of the other particles in the gas. The mean-free-time τ_{mfp} is defined to be the time for one collision to occur, i.e. for one particle to reside in the collision cylinder. Show that

$$\tau_{mfp} = \frac{1}{n\pi d^2 \bar{v}}$$

where n is the density of particles per unit volume. The corresponding distance traveled is $\lambda_{mfp} = \bar{v}\tau_{mfp} = 1/n\pi d^2$.

(b) In reality, all of the particles in the fluid are moving with a random velocity chosen from an ensemble with a probability density $p(\mathbf{v})$. This will ordinarily be Maxwellian, but we assume here only that the mean velocity is zero and that distinct particles have statistically independent and identically distributed velocities. Show that the *relative* velocity $\mathbf{v}-\mathbf{v}'$ of any pair of particles has the root-mean-square value $v_{rel} = \sqrt{\langle|\mathbf{v}-\mathbf{v}'|^2\rangle} = \sqrt{2}v_{rms}$ where $v_{rms}^2 = \int d^3v\ |\mathbf{v}|^2 p(\mathbf{v})$. The velocity $\bar{v}$ in the previous argument should be replaced by v_{rel}, whereas the distance traveled by the particle in a mean-free-time is $\lambda_{mfp} = v_{rms}\tau_{rms}$. Use these ideas to derive

$$\lambda_{mfp} = \frac{1}{n\sqrt{2}\pi d^2}.$$

(d) Estimate the mean-free-path length of air at atmospheric pressure $P = 1.1013 \times 10^5$ Pa and $T = 300°$K. Use the ideal gas law $P = nk_BT$ to obtain the density n and estimate $d = 2R$ where $R = 1.55 \times 10^{-10}$ m is the van der Waals radius of molecular nitrogen. Calculate the ratio $\epsilon = R/\lambda_{mfp}$, which is the relevant small parameter for the validity of the Boltzmann kinetic equation.

3. (a) Using the mass and momentum equations of the compressible Navier-Stokes system, derive the balance equation for kinetic energy:

$$\partial_t\left(\frac{1}{2}\rho v^2\right) + \boldsymbol{\nabla}\boldsymbol{\cdot}\left[\left(\frac{1}{2}\rho v^2 + P\right)\mathbf{v} + \mathbf{T}^{(1)}\boldsymbol{\cdot}\mathbf{v}\right] = -Q + P(\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{v})$$

where $Q = -\mathbf{T}^{(1)}\boldsymbol{:}\boldsymbol{\nabla}\mathbf{v} = 2\eta S^2 + \zeta D^2 \geq 0$ is the rate of kinetic energy dissipation per unit volume, with $D = \boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{v}$ the velocity-divergence/dilatation and $\mathbf{S} = \frac{1}{2}\left[\boldsymbol{\nabla}\mathbf{v} + (\boldsymbol{\nabla}\mathbf{v})^\top - (2/3)D\mathbf{I}\right]$ the velocity strain-rate tensor.

Hint: It is useful to show first that

$$\rho D_t\mathbf{v} + \boldsymbol{\nabla}P + \boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{T}^{(1)} = 0$$

with $D_t = \partial_t + \mathbf{v}\boldsymbol{\cdot}\boldsymbol{\nabla}$ the Lagrangian or material time-derivative.

(b) Use part (a) and conservation of total energy to derive the balance equation for internal energy density:

$$\partial_t u + \boldsymbol{\nabla}\boldsymbol{\cdot}\left[u\mathbf{v} + \mathbf{q}\right] = Q - P(\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{v}),$$

where $\mathbf{q} = -\kappa\boldsymbol{\nabla}T$ is the heat transport vector due to thermal conduction, or, alternatively,

$$D_t u + u(\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{v}) + \boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{q} = Q - P(\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{v}),$$

(c) Use the balance equation for conservation of mass to show that the velocity-divergence is given by

$$\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{v} = -D_t\rho/\rho = D_t\mathscr{V}/\mathscr{V}$$

where $\mathscr{V} = 1/n$ is the *specfic volume*, or volume per particle. Use this result to show that the pressure-dilatation term is given by

$$P(\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{v}) = -\frac{1}{\mathscr{V}}D_t W$$

where $D_t W \doteq -PD_t\mathscr{V}$ is the rate at which work is performed by the velocity field acting against pressure to compress the fluid.

(d) The entropy per volume $s(u,\rho)$ is a thermodynamic function which satisfies the first law of thermodynamics in the form

$$ds = (du - \mu\ d\rho)/T$$

where $T(u,\rho)$ is the absolute temperature and $\mu(u,\rho)$ is the chemical potential per mass. These thermodynamic state functions also satisfy the *homogeneous Gibbs relation*

$$u + P = sT + \mu\rho.$$

Using these standard facts from thermodynamics and the system of compressible Navier-Stokes equation, derive the balance equation for entropy density

$$\partial_t s + \boldsymbol{\nabla}\boldsymbol{\cdot}[s\mathbf{v} + \mathbf{q}/T] = \frac{Q}{T} + \frac{\kappa|\boldsymbol{\nabla}T|^2}{T^2} \geq 0.$$

This equation is a precise statement of the *second law of thermodynamics* for a single-component compressible fluid.

Hint: It is helpful to use the relation

$$D_t s = (1/T)D_t u - (\mu/T)D_t\rho$$

that follows from the first law of thermodynamics.

4. In this problem we are going to estimate in the atmospheric boundary layer (ABL) some characteristic velocities for small-scale turbulent fluctuations and for thermal fluctuations, and compare their magnitudes.

(a) As we shall discuss later, a rough estimate of the smallest length scale experiencing turbulent fluctuations is given by $\eta = (\nu^3/\varepsilon)^{1/4}$, where ν is the kinematic viscosity of the fluid and ε is the energy dissipation rate per unit mass. The corresponding turbulent velocity fluctuation at those scales is estimated by $v_\eta = (\nu\varepsilon)^{1/4}$. In the ABL at standard pressure and temperature $T = 300^\circ$K, reasonable numerical values are $\nu = 0.15\,\mathrm{cm}^2/\mathrm{sec}$ and $\varepsilon = 400\,\mathrm{cm}^2/\mathrm{sec}^3$. Calculate η and v_η.

(b) The individual molecules in a fluid in local thermodynamic equilibrium have velocities that are statistically independent with Gaussian (Maxwellian) distributions, whose mean is the local fluid velocity and whose standard deviation is roughly the sound speed c_s at the local fluid temperature. If one averages the velocities of individual molecules over a spherical region of radius ℓ, use the central limit theorem to explain why the fluctuation around this local average has the same mean but standard deviation $v_\ell^\theta = c_s/\sqrt{N_\ell}$, for $N_\ell = \frac{4}{3}n\pi\ell^3$ and n the number density.

(c) Use the results of part (a),(b) and the number density from Problem 2(d) to calculate v_η^θ at the length scale $\ell = \eta$. Note that the speed of sound in air at atmospheric pressure and $T = 300^\circ$K is about $c_s \doteq 347$ m/sec. How does this velocity of thermal fluctuations at that scale compare in magnitude with the turbulent velocity fluctuations v_η?