Homework #1

Problem 1.

$$\varepsilon_{tot} = \varepsilon_{oth} + \varepsilon_{turb}$$

$$\varepsilon_{oth} = \frac{7}{10}\varepsilon_{tot}, \quad \varepsilon_{turb} = \frac{3}{10}\varepsilon_{tot}$$

$$\varepsilon'_{turb} = \frac{0.3}{0.4}\varepsilon_{turb} = \frac{3}{4}\varepsilon_{turb} = \frac{9}{40}\varepsilon_{tot}$$

$$\varepsilon'_{tot} = \varepsilon_{oth} + \varepsilon'_{turb} = \left(\frac{7}{10} + \frac{9}{40}\right)\varepsilon_{tot} = \frac{37}{40}\varepsilon_{tot}$$

$$\Longrightarrow \quad \varepsilon'_{tot} = (0.925)\,\varepsilon_{tot}$$

or

$$\frac{\varepsilon_{tot} - \varepsilon'_{tot}}{\varepsilon_{tot}} = 0.075 = 7.5\%$$

Problem 2.

$$\frac{dE}{dt} = -\varepsilon = -\frac{u^3}{L}$$

Since $E = \frac{3}{2}u^2 \Longrightarrow u = \left(\frac{2}{3}E\right)^{1/2}$, therefore

$$\frac{dE}{dt} = -\left(\frac{2}{3}\right)^{3/2}\frac{E^{3/2}}{L} = -2K\,E^{3/2}$$

with

$$K = \left(\frac{2}{3}\right)^{3/2}\frac{1}{2L}$$

Integrating

$$-\frac{1}{2}\frac{dE}{E^{3/2}} = K dt$$

gives $E^{-1/2} - E_0^{-1/2} = Kt$ or

$$E = \left[E_0^{-1/2} + Kt \right]^{-2}$$

Thus, for times $t \gg 1/KE_0^{1/2}$ the decay is power-law $\propto t^{-2}$.

The final period of decay sets in when $t = t_*$ specified by $Re = \frac{uL}{\nu} = 10$. Setting

$$\frac{c\nu u^2(t_*)}{L^2} = \frac{u^3(t_*)}{L}$$

$$\Longrightarrow c = \frac{Lu(t_*)}{\nu} = 10$$

To obtain the Reynolds number, use $u = \left(\frac{2}{3}E\right)^{1/2}$ ~

$$u = \left(\frac{2}{3}\right)^{1/2} \left[E_0^{-1/2} + K \cdot t \right]^{-1}$$

$$= \left[u_0^{-1} + \left(\frac{3}{2}\right)^{1/2} K \cdot t \right]^{-1}$$

This gives

$$Re = \frac{uL}{\nu} = \frac{L}{\nu}\left[u_0^{-1} + \left(\frac{3}{2}\right)^{1/2} kt\right]^{-1}$$

$$= \left[Re_0^{-1} + \frac{1}{3}\frac{\nu t}{L^2}\right]^{-1}$$ from the definition of k.

and

$$\frac{1}{10} = \frac{1}{Re} = Re_0^{-1} + \frac{\nu}{3L^2} t_*$$

$$\Rightarrow t_* = \frac{3L^2}{\nu}\left(\frac{1}{10} - Re_0^{-1}\right)$$

$$\longrightarrow \frac{3L^2}{10\nu} \quad \text{as} \quad Re_0 \to \infty$$

For

$$Re_0 = \frac{u_0 L}{\nu} = \frac{\left(1\,\frac{m}{sec}\right)(1\,m)}{15 \times 10^{-6}\ m^2/sec} = \left(15 \times 10^{-6}\right)^{-1}$$

$$t_* = \frac{3L^2}{\nu}\left(\frac{1}{10} - 1.5 \times 10^{-5}\right)$$

$$= \frac{3(1m)^2}{15 \times 10^{-6}\ m^2/sec}\,(0.099985)$$

$$= 19{,}997 \text{ secs} \quad \text{or} \quad 5.555 \text{ hrs}$$

For $t > t_*$, the decay follows

$$\frac{dE}{dt} = -\frac{10\nu}{L^2} u^2$$

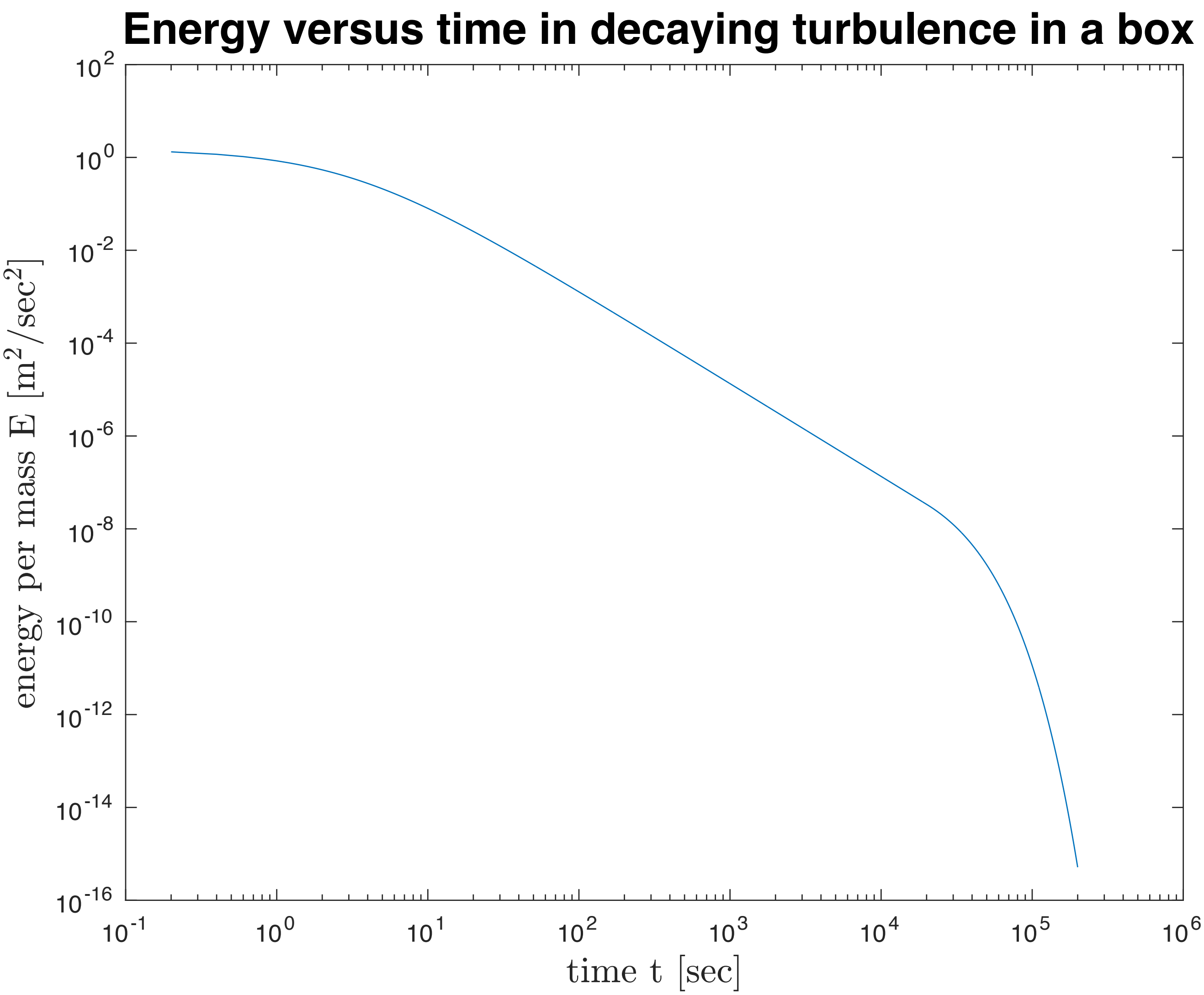
$$= -\frac{10\nu}{L^2} \cdot \frac{2}{3}\left(\frac{3}{2}u^2\right) = \frac{-20\nu}{3L^2} E$$

$$\Longrightarrow \quad \underline{E = E(t_*)\, e^{-20\nu(t-t_*)/3L^2}}$$

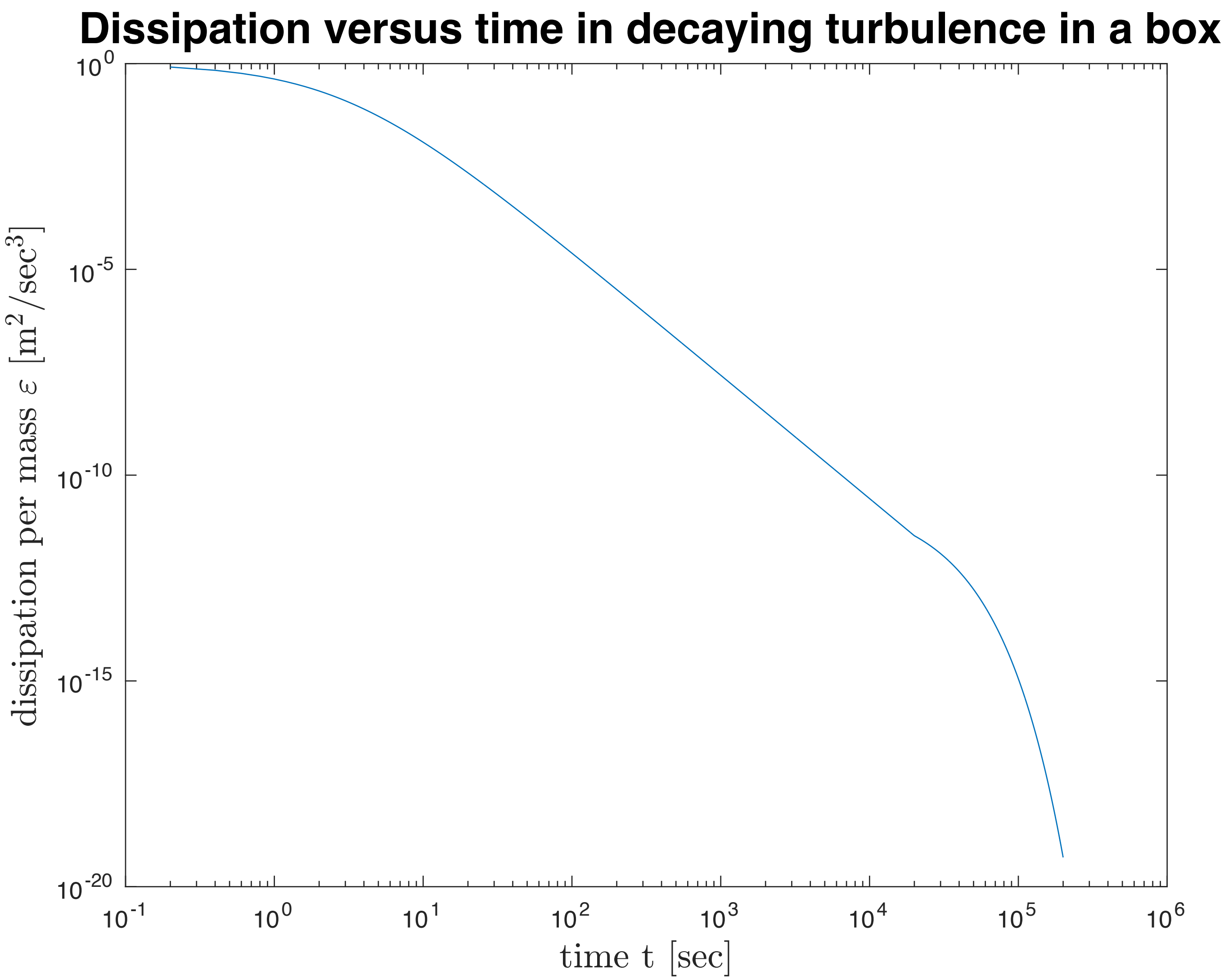
with $E(t_*) = \left[E_0^{-1/2} + K t_* \right]^{-2}$.

Thus, in the final period of decay, the energy decreases exponentially.

<u>NOTE</u>: These results depend <u>crucially</u> on the assumption that L is independent of time! In general, this is not true in decaying turbulence until the "integral length-scale" of the turbulence grows to the size of the fluid domain (= box-size).



Energy versus time in decaying turbulence in a box
10^{2}
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
10^{-16}
energy per mass E $[\mathrm{m}^2/\mathrm{sec}^2]$
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}
10^{5}
10^{6}
time t [sec]


Dissipation versus time in decaying turbulence in a box
dissipation per mass ε [m^2/sec^3]
10^0
10^{-5}
10^{-10}
10^{-15}
10^{-20}
10^{-1}
10^0
10^1
10^2
10^3
10^4
10^5
10^6
time t [sec]

Problem 3

The Prandtl - Kármán relation

$$\frac{U_0}{u_*} = 2.5 \ln\left(\frac{u_* r}{\nu}\right) + 6$$

$$= 2.5 \ln\left(\frac{U_0 r}{\nu} \cdot \frac{u_*}{U_0}\right) + 6$$

gives for the factor $F = U_0 / u_*$

$$F = 2.5 \ln\left(\frac{Re}{F}\right) + 6 \qquad (*)$$

$$= 2.5 \ln(Re) - 2.5 \ln F + 6 \doteq 2.5 \ln(Re)$$

Then from $(*)$ we obtain

$$\frac{F-6}{2.5} + \ln F = \ln Re$$

$$\Longrightarrow \quad Re = F \exp\left(\frac{F-6}{2.5}\right) \qquad \frac{6}{2.5} = 2.4$$

$$\Longrightarrow \quad e^{2.4} Re = F \exp\left(\frac{F}{2.5}\right)$$

$$\Longrightarrow \quad \frac{e^{2.4} Re}{2.5} = \frac{F}{2.5} \exp\left(\frac{F}{2.5}\right)$$

$$\Longrightarrow \quad \frac{F}{2.5} = W\left(\frac{e^{2.4} Re}{2.5}\right)$$

We have introduced here the Lambert W-function

$$we^{w} = z \iff w = W(z)$$

Note this may be evaluated in Matlab with the command lambertw. Thus,

$$F = 2.5\, W\left(\frac{e^{2.4\,Re}}{2.5}\right)$$

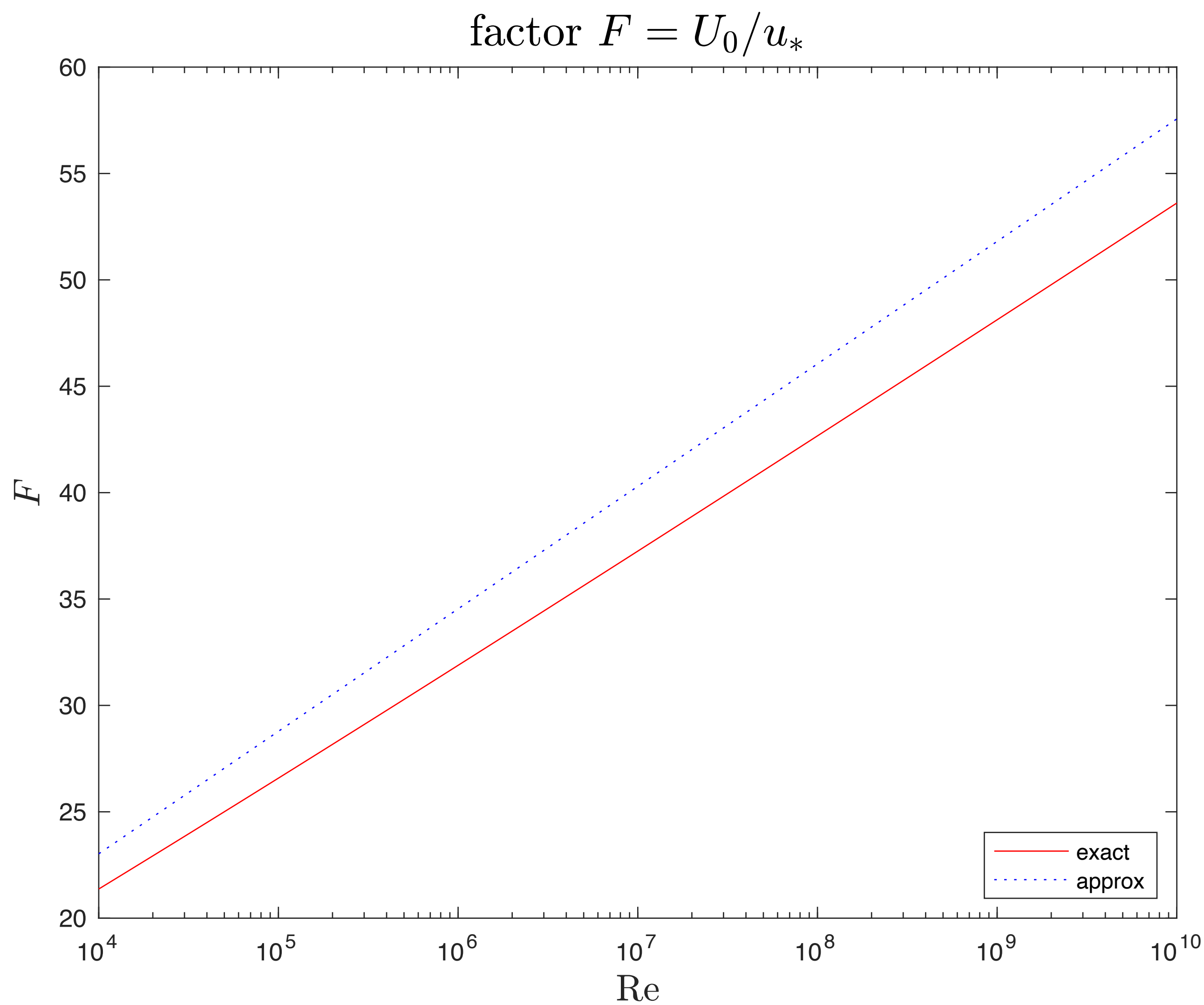
Alternatively, one may solve the nonlinear equation $F + 2.5 \ln F = 2.5 \ln(Re) + 6$ by numerical algorithms such as Newton's method to obtain $F = F(Re)$.

Finally, from

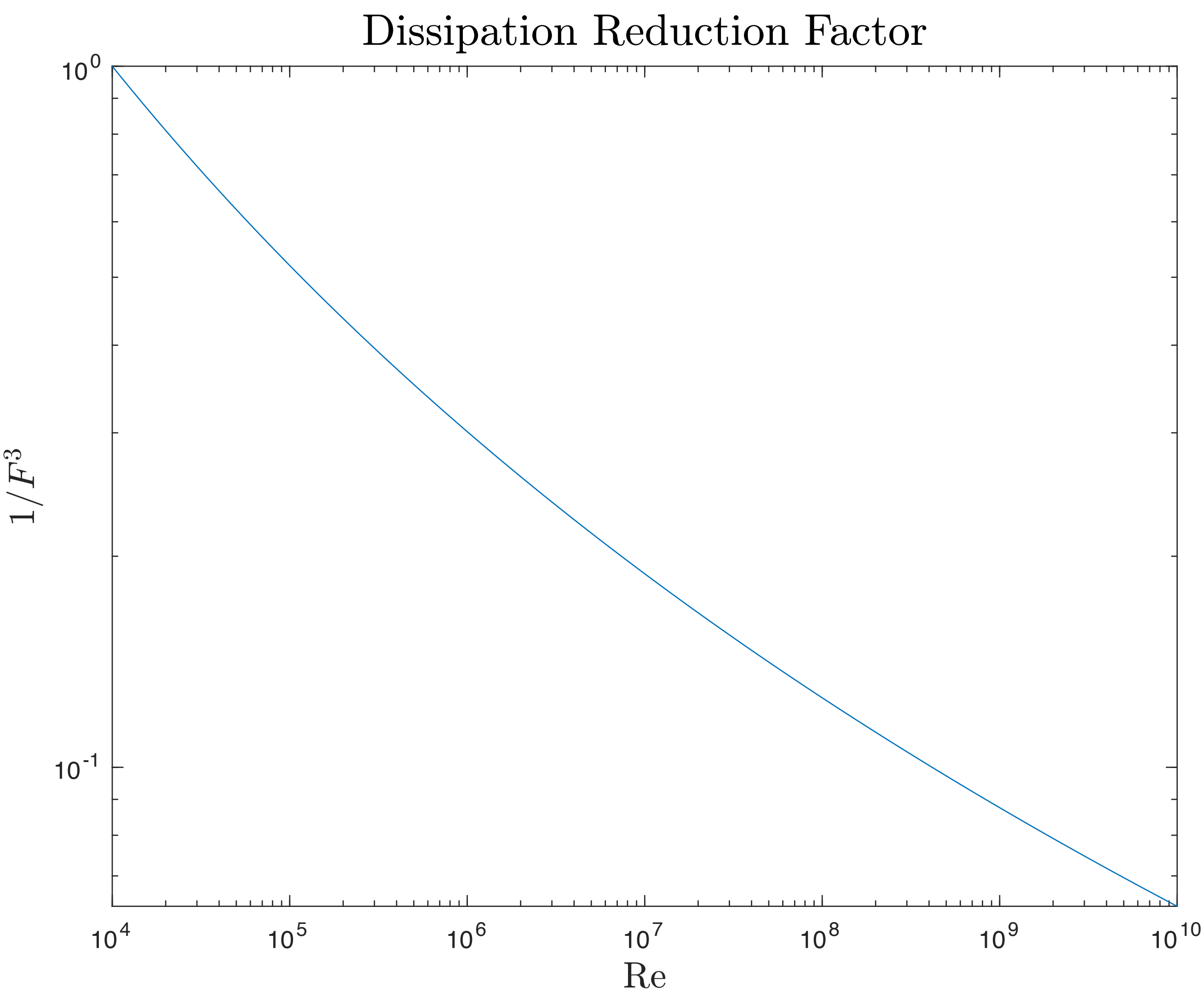
$$\varepsilon \sim \frac{u_*^3}{r} \sim \frac{U_0^3}{r} F^{-3}$$

we see that bulk dissipation with U_0, r fixed decreases proportional to F^{-3} with increasing Re,

On the next pages we plot both the factors F and F^{-3} versus Re over the range of bulk Reynolds numbers from 10^4 to 10^{10}.



factor $F = U_0/u_*$
60
55
50
45
40
35
30
25
20
F
10^4
10^5
10^6
10^7
10^8
10^9
10^{10}
Re
exact
approx


Dissipation Reduction Factor
10^0
10^{-1}
$1/F^3$
10^4
10^5
10^6
10^7
10^8
10^9
10^{10}
Re

For Reynolds number increasing from $Re = 10^4$ to $Re = 10^{10}$, the reduction in energy dissipation is

$$1/F^3 \cong 0.06335.$$

Although the Reynolds number increases by a factor of <u>one million</u>, the dissipation decreases by a factor of only about <u>16</u>! Such a slow decrease is practically almost no different than a dissipation independent of Re. Such a slow decrease is also seen in the experiments of Cadot et al. (1997) [together perhaps with a dissipation contribution from the bulk of the flow which is truly independent of Re!].

Problem 4.

$$D = 5\ mm^2/sec$$

$$= 5 \times 10^{-6}\ m^2/sec$$

$$T = \frac{L^2}{D} = \frac{(5m)^2}{5 \times 10^{-6}\ m^2/sec} = 5 \times 10^6\ sec$$

$$T = \frac{5 \times 10^6}{3.6 \times 10^3}\ hr = 1388.\overline{8}\ hr$$

$$T = \frac{1388.\overline{8}}{24}\ days = 57.9\ days\ !$$

Problem 5. With $L \cong 250\,m$, $u_{rms} \cong 1\,m/sec$

$$\varepsilon \sim \frac{u_{rms}^3}{L} \doteq 4 \times 10^{-3} \frac{m^2}{sec^2}.$$

Thus,

$$\rho\varepsilon \doteq 5 \times 10^{-3} \frac{watt}{m^3}$$

The total volume of the cloud eddy is

$$V \cong \frac{4}{3}\pi R^3 \doteq 6.545 \times 10^7\, m^3$$

Thus, the total power dissipated in this one large-eddy of the cloud is

$$P = \rho\varepsilon V \doteq 327.25\ kwatt.$$

The total power supplied by sunlight (on a clear day) to a similar area of land is

$$P = \frac{1\,kw}{m^2}\,\pi(250\,m)^2 \doteq 2 \times 10^5\ kwatt$$

One can see that only a small fraction of this energy must be transferred into the atmosphere by convection in order to power the cumulus cloud!

On the other hand,

$$\underline{1} \text{ horsepower} \doteq 0.7355 \text{ kw}$$

so that

$$300 \text{ horsepower} \doteq 221 \text{ kw},$$

which is somewhat less than the power dissipated in the largest eddy of a cumulus cloud.