The Local Structure of Turbulence in Incompressible Viscous Fluid for Very Large Reynolds Numbers
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The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers†

By A. N. Kolmogorov

§1. We shall denote by

$$u_\alpha(P) = u_\alpha(x_1, x_2, x_3, t), \quad \alpha = 1, 2, 3,$$

the components of velocity at the moment t at the point with rectangular cartesian coordinates x_1, x_2, x_3. In considering the turbulence it is natural to assume the components of the velocity $u_\alpha(P)$ at every point $P = (x_1, x_2, x_3, t)$ of the considered domain G of the four-dimensional space (x_1, x_2, x_3, t) are *random variables* in the sense of the theory of probabilities (cf. for this approach to the problem Millionshtchikov (1939)).

Denoting by $\bar{A}$ the mathematical expectation of the random variable A we suppose that

$$\bar{u}_\alpha^2 \quad \text{and} \quad \overline{(\mathrm{d}u_\alpha/\mathrm{d}x_\beta)^2}$$

are finite and bounded in every bounded subdomain of the domain G.

Introduce in the four-dimensional space (x_1, x_2, x_3, t) new coordinates

$$\left.\begin{aligned} y_\alpha &= x_\alpha - x_\alpha^{(0)} - u_\alpha(P^{(0)})(t - t^{(0)}), \\ s &= t - t^{(0)}, \end{aligned}\right\} \tag{1}$$

where

$$P^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, t)$$

is a certain fixed point from the domain G. Observe that the coordinates y_α of any point P depend on the random variables $u_\alpha(P^{(0)})$ and hence are themselves random variables. The velocity components in the new coordinates are

$$w_\alpha(P) = u_\alpha(P) - u_\alpha(P^{(0)}). \tag{2}$$

Suppose that for some fixed values of $u_\alpha(P^{(0)})$ the points $P^{(k)}$, $k = 1, 2, \ldots, n$, having the coordinates $y_\alpha^{(k)}$ and $s^{(k)}$ in the coordinate system (1), are situated in the domain G. Then we may define a $3n$-dimensional distribution law of probabilities F_n for the quantities

$$w_\alpha^{(k)} = w_\alpha(P^{(k)}), \quad \alpha = 1, 2, 3; \quad k = 1, 2, \ldots, n,$$

where

$$u_\alpha^{(0)} = u_\alpha(P^{(0)})$$

are given.

Generally speaking, the distribution law F_n depends on the parameters $x_\alpha^{(0)}$, $t^{(0)}$, $u_\alpha^{(0)}$, $y_\alpha^{(k)}$, $s^{(k)}$.

Definition 1. *The turbulence is called locally homogeneous in the domain G, if for every fixed n, $y_\alpha^{(k)}$ and $s^{(k)}$, the distribution law F_n is independent from $x_\alpha^{(0)}$, $t^{(0)}$ and $u_\alpha^{(0)}$ as long as all points $P^{(k)}$ are situated in G.*

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Definition 2. *The turbulence is called locally isotropic in the domain G, if it is homogeneous and if, besides, the distribution laws mentioned in Definition 1 are invariant with respect to rotations and reflections of the original system of coordinate axes* (x_1, x_2, x_3).

In comparison with the notion of *isotropic turbulence* introduced by Taylor (1935) our definition of locally isotropic turbulence is *narrower* in the sense that in our definition we demand the independence of the distribution law F_n from $t^{(0)}$, i.e. steadiness in time, and is *wider* in the sense that restrictions are imposed only on the distribution laws of differences of velocities and not on the velocities themselves.

§2. The hypothesis of isotropy in the sense of Taylor is experimentally quite well confirmed in the case of turbulence caused by passing of a flow through a grid (cf. (3)). In the majority of other cases interesting from the practical point of view it may be considered only as a rather far approximation of reality even for small domains G and very large Reynolds numbers.

On the other hand we think it rather likely that in an arbitrary turbulent flow with a sufficiently large Reynolds number†

$$R = LU/\nu$$

the hypothesis of local isotropy is realized with good approximation in sufficiently small domains G of the four-dimensional space (x_1, x_2, x_3, t) not lying near the boundary of the flow or its other singularities. By a 'small domain' we mean here a domain, whose linear dimensions are small in comparison with L and time dimensions – in comparison with

$$T = U/L.$$

It is natural that in so general and somewhat indefinite a formulation the just advanced proposition cannot be rigorously proved.‡ In order to make its ex-

† Here L and U denote the typical length and velocity for the flow in the whole.

‡ We may indicate here only certain general considerations speaking for the advanced hypothesis. For very large R the turbulent flow may be thought of in the following way: on the averaged flow (characterized by the mathematical expectations $\bar{u}_\sigma$) are superposed the 'pulsations of the first order' consisting in disorderly displacements of separate fluid volumes, one with respect to another, of diameters of the order of magnitude $l^{(1)} = l$ (where l is the Prandtl's mixing path); the order of magnitude of velocities of these relative velocities we denote by $v^{(1)}$. The pulsations of the first order are for very large R in their turn unsteady, and on them are superposed the pulsations of the second order with mixing path $l^{(2)} < l^{(1)}$ and relative velocities $v^{(2)} < v^{(1)}$; such a process of successive refinement of turbulent pulsations may be carried until for the pulsations of some sufficiently large order n the Reynolds number

$$R^{(n)} = l^{(n)} v^{(n)}/\nu$$

becomes so small that the effect of viscosity on the pulsations of the order n finally prevents the formation of pulsations of the order $n+1$.

From the energetical point of view it is natural to imagine the process of turbulent mixing in the following way: the pulsations of the first order absorb the energy of the motion and pass it over successively to pulsations of higher orders. The energy of the finest pulsations is dispersed in the energy of heat due to viscosity.

In virtue of the chaotical mechanisms of the translation of motion from the pulsations of lower orders to the pulsations of higher orders, it is natural to assume that in domains of the space, whose dimensions are small in comparison with $l^{(1)}$, the fine pulsations of the higher orders are subjected to approximately space-isotropic statistical régime. Within small time-intervals it is natural to consider this régime approximately steady even in the case, when the flow in the whole is not steady.

Since for very large R the differences

$$w_\alpha(P) = u_\alpha(P) - u_\alpha(P^{(0)})$$

perimental verification possible in particular cases we indicate here a number of consequences of the hypothesis of local isotropy.

§3. Denoting by y the vector with components y_1, y_2, y_3, we consider the random variables

$$w_\alpha(y) = w_\alpha(y_1, y_2, y_3) = u_\alpha(x_1+y_1, x_2+y_2, x_3+y_3, t) - u_\alpha(x_1, x_2, x_3, t). \tag{3}$$

In virtue of the assumed local isotropy their distribution laws are independent from x_1, x_2, x_3 and t. From the first moments of the quantities $w_\alpha(y)$ it follows from the local isotropy that

$$\overline{w_\alpha(y)} = 0. \tag{4}$$

We proceed therefore to the consideration of the second moments†

$$B_{\alpha\beta}(y^{(1)}, y^{(2)}) = \overline{w_\alpha(y^{(1)}) w_\beta(y^{(2)})}. \tag{5}$$

From the local isotropy follows that

$$B_{\alpha\beta}(y^{(1)}, y^{(2)}) = \tfrac{1}{2}[B_{\alpha\beta}(y^{(1)}, y^{(1)}) + B_{\alpha\beta}(y^{(2)}, y^{(2)}) - B_{\alpha\beta}(y^{(2)} - y^{(1)}, y^{(2)} - y^{(1)})]. \tag{6}$$

In virtue of this formula we may confine ourselves to the second moments of the form $B_{\alpha\beta}(y, y)$. For them

$$B_{\alpha\beta}(y, y) = \bar{B}(r) \cos\theta_\alpha \cos\theta_\beta + \delta_{\alpha\beta} B_{nn}(r), \tag{7}$$

where

$$r^2 = y_1^2 + y_2^2 + y_3^2, \quad y_\alpha = r\cos\theta_\alpha, \quad \delta_{\alpha\beta} = 0 \quad \text{for} \quad \alpha \neq \beta, \quad \delta_{\alpha\beta} = 1 \quad \text{for} \quad \alpha = \beta,$$

$$\bar{B}(r) = B_{dd}(r) - B_{nn}(r), \tag{8}$$

$$\left.\begin{aligned} B_{dd}(r) &= \overline{[w_1(r,0,0)]^2}, \\ B_{nn}(r) &= \overline{[w_2(r,0,0)]^2}. \end{aligned}\right\} \tag{9}$$

For $r = 0$ we have

$$B_{dd}(0) = B_{nn}(0) = \frac{\partial}{\partial r} B_{dd}(0) = \frac{\partial}{\partial r} B_{nn}(0) = 0, \tag{10}$$

$$\left.\begin{aligned} \frac{\partial^2}{\partial r^2} B_{dd}(0) &= 2\overline{\left(\frac{\partial w_1}{\partial y_1}\right)^2} = 2a^2, \\ \frac{\partial^2}{\partial r^2} B_{nn}(0) &= 2\overline{\left(\frac{\partial w_2}{\partial y_1}\right)^2} = 2a_n^2. \end{aligned}\right\} \tag{11}$$

The formulae (6)–(11) were obtained without use of the assumption of incompressibility of the fluid. From this assumption follows the equation

$$r\,\partial B_{dd}/\partial r = -2\bar{B}, \tag{12}$$

enabling us to express B_{nn} through B_{dd}. From (12) and (11) follows that

$$a_n^2 = 2a^2. \tag{13}$$

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of the velocity components in neighbouring points P and $P^{(0)}$ of the four-dimensional space (x_1, x_2, x_3, t) are determined nearly exclusively by pulsations of higher orders, the scheme just exposed leads us to the hypothesis of local isotropy in small domains G in the sense of Definitions 1 and 2.

† All results of §3 are quite similar to that obtained in (1), (2) and (4) for the case of isotropic turbulence in the sense of Taylor.

It is, further, easy to calculate that (assuming the incompressibility) the average dispersion of energy in unit of time per unit of mass is equal to

$$\bar{\epsilon} = \nu \left\{ 2\overline{\left(\frac{\partial w_1}{\partial y_1}\right)^2} + 2\overline{\left(\frac{\partial w_2}{\partial y_2}\right)^2} + 2\overline{\left(\frac{\partial w_3}{\partial y_s}\right)^2} + \overline{\left(\frac{\partial w_2}{\partial y_1} + \frac{\partial w_1}{\partial y_2}\right)^2} \right.$$
$$\left. + \overline{\left(\frac{\partial w_3}{\partial y_2} + \frac{\partial w_2}{\partial y_3}\right)^2} + \overline{\left(\frac{\partial w_1}{\partial y_3} + \frac{\partial w_3}{\partial y_1}\right)^2} \right\} = 15\nu a^2. \quad (14)$$

§4. Consider the transformation of coordinates

$$y'_\alpha = y_\alpha/\eta, \quad s' = s/\sigma. \quad (15)$$

The velocities, the kinematical viscosity and the average dispersion of energy are expressed in the new system of coordinates by the following formulae:

$$w'_\alpha = w_\alpha \sigma/\eta, \quad \nu' = \nu\sigma/\eta^2, \quad \bar{\epsilon}' = \bar{\epsilon}\sigma^3/\eta^2. \quad (16)$$

We introduce now the following hypothesis.

The first hypothesis of similarity. *For the locally isotropic turbulence the distributions F_n are uniquely determined by the quantities ν and $\bar{\epsilon}$.*

The transformation of coordinates (15) leads for

$$\eta = \lambda = \sqrt{(\nu/\alpha)} = \nu^{\frac{3}{4}}/\bar{\epsilon}^{\frac{1}{4}} \quad (17)$$

and

$$\sigma = 1/\alpha = \sqrt{(\nu/\bar{\epsilon})} \quad (18)$$

to the quantities $\nu' = 1$, $\bar{\epsilon}' = 1$.

In virtue of the accepted hypothesis of similarity the corresponding function

$$B'_{dd}(r') = \beta_{dd}(r') \quad (19)$$

must be the same for all cases of locally isotropic turbulence. The formula

$$B_{dd}(r) = \sqrt{(\nu\bar{\epsilon})}\,\beta_{dd}(r/\lambda) \quad (20)$$

shows in combination with the already deduced that in the case of locally isotropic turbulence the second moments $B_{\alpha\beta}(y^{(1)}, y^{(2)})$ are uniquely expressed through ν, $\bar{\epsilon}$ and the universal function β_{dd}.

§5. To determine the behaviour of the function $\beta_{dd}(r')$ for large r' we introduce another hypothesis.

The second hypothesis of similarity.† *If the moduli of the vectors $y^{(k)}$ and of their differences $y^{(k)} - y^{(k')}$ (where $k \neq k'$) are large in comparison with λ, then the distribution laws F_n are uniquely determined by the quantity $\bar{\epsilon}$ and do not depend on ν.*

Put

$$y''_\alpha = y'_\alpha/k^3, \quad s'' = s'/k^2, \quad (21)$$

where y'_α and s' are determined in accordance with the formulae (15), (17) and (18). Since for every k $\bar{\epsilon}' = \bar{\epsilon}'' = 1$, for r' large in comparison with $\lambda' = 1$ we have in virtue of the accepted hypothesis

$$B''_{dd}(r'') \sim B'_{dd}(r'') = \beta_{dd}(r'/k^3).$$

† In terms of the schematical representation of turbulence developed in the footnote ‡, λ is the scale of the finest pulsations, whose energy is directly dispersed into heat energy due to viscosity. The sense of the second hypothesis of similarity consists in that the mechanism of translation of energy from larger pulsations to the finer ones is for pulsations of intermediate orders, for which $l^{(k)}$ is large in comparison with λ, independent from viscosity.

On the other hand, from the formula (20) follows that

$$B''_{dd}(r'') = k^{-2}B'_{dd}(r') = k^{-2}\beta_{dd}(r').$$

Thus for large r'

$$\beta_{dd}(r'/k^3) \sim k^{-2}\beta_{dd}(r'),$$

whence

$$\beta_{dd}(r') \sim C(r')^{\frac{2}{3}}, \tag{22}$$

where C is an absolute constant. In virtue of (17), (20) and (22) we have for r large in comparison with λ

$$B_{dd}(r) \sim C\bar{\epsilon}^{\frac{2}{3}}r^{\frac{2}{3}}. \tag{23}$$

From (23) and (12) it is easy to deduce that for r large in comparison with λ

$$B_{nn}(r) \sim \tfrac{4}{3}B_{dd}(r). \tag{24}$$

As regards the last formula, observe that for r *small* in comparison with λ in virtue of (13) holds the relation

$$B_{nn}(r) \sim 2B_{dd}(r). \tag{25}$$

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