order of the dissipation wave number $k_{D,1}$. The relative error is

$$r \sim E^{err}(k_{D,1})/E(k_{D,1}) \sim k_{D,1}/k_2 \ . \tag{3.8}$$

We now assume exponential amplification of the error with a characteristic time equal to the turn-over time l_1/v_1 in the enstrophy inertial range (roughly wave number independent); we find that the relative error has become of order unity after a time $\sim t_1 \log(1/r) \sim t_1 \log(k_2/k_{D,1})$. To this we must add the time for errors to migrate along the enstrophy inertial range from $k_{D,1}$ to k_1 which is $\sim t_1 \log(k_{D,1}/k_1)$. Hence the total predictability time is $\sim t_1 \log(k_2/k_1)$. In other words, thanks to the spectral gap, the predictability is the same as if we had a 2-D flow with full resolution of all scales down to the integral scale of the 3-D turbulence. The increased predictability can be rather important (days?) if a spectral gap exists at scales ~ 100 km. This, of course, is debatable. It could be that the gap in the energy spectrum is filled by rare but violent meteorological events and does not exist in the mean; some increase in the predictability is then nevertheless expected.

Finally, we observe that coherent structures may play an important part in the dynamics of atmospheric turbulence. If this is so, predictability estimates based on turbulence phenomenology (*à la* Kolmogorov) may be very misleading.

* * *

We have greatly benefitted from discussions with C. GLOAGUEN (two-component shell model) and with J. C. ANDRÉ, M. LESIEUR and O. THUAL (spectral gap and atmospheric predictability).

APPENDIX

On the singularity structure of fully developed turbulence.

with

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A simple way of explaining power law structure function is to invoke singularities of the Euler equations considered as limit of the Navier-Stokes equations as the viscosity tends to zero. For Burgers' equation we know that such singularities exist (shocks) and that they provide the required explanation of scaling. For the 2-D Euler equations we know that singularities do not exist (see, *e.g.*, ref. [26] and references therein). For the 3-D Euler equations the numerical evidence is inconclusive [26, 31]. MANDELBROT [24, 25] and others [20] have considered models with singularities concentrated on a set $\subset \boldsymbol{R}^3$ having noninteger (fractal) Hausdorff dimension. We shall here show that the data suggest the existence of a hierarchy of such sets (a « multifractal »).

Since the Navier-Stokes equations (in the zero-viscosity limit) are invariant under the group of scaling transformations (defined in eq. (2.2)) for *any* value of h, singularities of arbitrary exponents (and mixtures thereof) are consistent with the equations. Specifically, we start with a definition, the velocity field at a given time $v(x)$ is said to have a singularity of order $h > 0$ at the point x if

$$\overline{\lim_{\boldsymbol{x}\to\boldsymbol{y}}}\, |\boldsymbol{v}(\boldsymbol{x}) - \boldsymbol{v}(\boldsymbol{y})|/|\boldsymbol{x} - \boldsymbol{y}|^h \neq 0 \,. \tag{A.1}$$

For negative h eq. (A.1) is modified by not subtracting $\boldsymbol{v}(\boldsymbol{y})$.

We call $S(h)$ the set of points for which the velocity field has a singularity of order h. It is obvious that

$$S(h') \supset S(h) \qquad \text{if } h' > h \,. \tag{A.2}$$

Roughly speaking, $S(h)$ is the region where the velocity field is not an Hölder function of order h. We denote by $d(h)$ the Hausdorff dimension of $S(h)$ (see ref. [34] and [50] for definitions). It follows from eq. (A.2) that $d'(h) > 0$; we also make the concavity assumption $d''(h) < 0$.

If such singularities exist, then, in the fully developed turbulence regime, $d(h)$ has a nontrivial dependence on h: different kind of singularities are associated with sets having different Hausdorff dimensions. Note that the opposite phenomenon happens for the solutions of stochastic differential equations with white noise (like those studied in Jona-Lasinio's contribution to this volume): there the one-dimensional trajectories are (with probability one) Hölder functions of order $\frac{1}{2}$, so that

$$d(h) = \theta(h - \tfrac{1}{2}) \qquad (\theta = \text{step function}) \,. \tag{A.3}$$

It is useful to connect the function $d(h)$ with the exponents ζ_p introduced in eq. (2.4) which control the asymptotic behaviour of the longitudinal structure functions. We can try to rephrase the previous statements on the Hausdorff dimensions of $S(h)$ by saying that the probability of having $|\boldsymbol{v}(\boldsymbol{x}) - \boldsymbol{v}(\boldsymbol{y})|$ of order $|\boldsymbol{x} - \boldsymbol{y}|^h$ goes to zero like $|\boldsymbol{x} - \boldsymbol{y}|^{3-d(h)}$ when $|\boldsymbol{x} - \boldsymbol{y}| \to 0$. We thus arrive to the following integral representation for the moments:

$$\langle(\delta v(l))^p\rangle \sim \int \mathrm{d}\mu(h)\, l^{(ph+3-d(h))} \,, \tag{A.4}$$

where $\mathrm{d}\mu(h)$ is a measure concentrated on the region where $d(h) > 0$.

In the K 41 [18] picture and in the β-model [20], we have, respectively,

$$\begin{cases} \zeta_p = p/3 \,, & \\ \zeta_p = \lambda(p-3) + 1 & (\lambda < \frac{1}{3}) \,. \end{cases} \tag{A.5}$$

Consequently we have, respectively,

$$\begin{cases} d(h) = 3\theta(h - \frac{1}{3}) \,, & \text{K 41} \,, \\ d(h) = (2 + 3\lambda)\theta(h - \lambda) \,, & \beta\text{-model} \,. \end{cases} \tag{A.6}$$

In more sophisticated models, and also in actual turbulence, according to ref. [29], ζ_p is a nonlinear function of p. Evaluating the integral (A.4) using the saddle point method, we easily find

$$\zeta_p = \min_h [ph + 3 - d(h)] . \tag{A.7}$$

We have thus found that ζ_p is the Legendre transform (see ref. [51], sect. 14) of the codimension $(c(h) \equiv 3 - d(h))$ of the set $S(h)$. This is assuring that the convexity properties of ζ_p are automatically preserved by eq. (A.7).

If eqs. (A.4) and (A.7) are correct, the dimensions $d(h)$ are experimentally well-defined quantities: they can be extracted from the ζ_p's by using the inverse Legendre transform

$$d(h) = 3 - \min_p (\zeta_p - ph) . \tag{A.8}$$

We shall not try to do this using the data displayed in table I, although this is clearly possible, at least in the range of h for which the value of p minimizing eq. (8) falls in the experimentally observed interval: it is, however, likely that $d(h)$ will not be a step function because ζ_p appears to significantly deviate from a linear function of p. The function $d(h)$ is thus nontrivial and singularities of different kinds, if they exist, are concentrated on sets having different Hausdorff dimensions.

The function $d(h)$ (or, equivalently, ζ_p) has a clear dynamical meaning because it contains most of the relevant information on the scaling laws for fully developed turbulence. It would be rather important to measure accurately $d(h)$ and to find good evidence for its universality, *i.e.* its independence on the initial conditions and on all the other parameters which should become irrelevant in the fully developed turbulence regime.

If the multifractal model is basically correct, accurate measurements of the ζ_p's may be quite difficult. Indeed, the structure functions are a mixture of power laws (eq. (A.4)), so that very small scales (*i.e.* very high Reynolds numbers) may be needed before the contribution with the smallest exponent clearly dominates; where exactly this happens depends on the distribution $d\mu(h)$.

Note that consistency of the multifractal model with the data is by no means evidence for real singularities of the Euler equations. There is certainly more than one way to obtain scaling, otherwise scaling would not be observed in two dimensions, where singularities are ruled out [26].

We note two interesting consequences of the inversion formula (A.8). First, if ζ_p vanishes for $p \to 0$, then the weakest singularities, which has the exponent ζ_0', are space filling ($d = 3$). It is clearly of interest to measure ζ_p for small noninteger p's. Second, the multifractal model is not completely consistent with Kolmogorov's [23] lognormal model for which $\zeta_p = p/3 + \mu p(3-p)/18$ (μ, if it exists, is somewhere between 0.2 and 0.5; see ref. [29]). Indeed, with this choice of ζ_p we find from eq. (A.8) that beyond $p_{\max} = 9(2/3\mu)^{\frac{1}{2}}$ a negative dimension is obtained. Accurate measurements of very-high-order structure functions are required to test for a possible inconsistency of the multifractal model.

Finally, one may wonder how the above «multifractal» model relates to the models of ref. [20, 25, 34, 52]. In Mandelbrot's [25, 34, 52] probabilistic models for the dissipation a random weighting factor W appears at each stage of the cascade. The case when W has a binomial distribution («absolute curdling») corresponds to a single fractal in our approach (it is also equivalent to the β-model). For more general W-distributions (« weighted curdling ») one obtains exponents ζ_p that depend nonlinearly on p like in the multifractal model. There is a single fractal for the energy dissipation, but it is conceivable that other fractals will be uncovered by investigating all possible singularities of the dissipation. Still the multifractal model appears to be somewhat more restrictive than Mandelbrot's weighted-curdling model which does include the lognormal case.

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