Thermal Agitation and Turbulence

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This paper investigates the nature and the possible effects of thermal agitation in ordinary fluids and in plasmas. The spectra of the velocity fluctuations and of the magnetic field fluctuations are obtained under the assumption of incompressibility. The amplification of thermal fluctuations in shear flows is studied by referring to the theory of hydrodynamic stability. The results suggest that amplified thermal agitation may cause turbulent transition in many common situations. The amplification of thermal fluctuations may also be important in a fully developed turbulent flow.

I. Introduction

The object of this study is to clarify the relation between turbulence and thermal agitation. By turbulence, we mean the random motion of a fluid. It is generally regarded as a macroscopic phenomenon. By thermal agitation we mean the directly observable aspects of thermal motion which have a random character.

To be more specific, we must distinguish the various manifestations of the thermal motion of molecules and atoms. It is well known that

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molecular processes are the cause of pressure forces and viscous stresses. These quantities are generally regarded as macroscopic and free from fluctuations, within certain limits. It is also well known that molecular processes are the cause of fluctuating forces. For instance, a particle immersed in a fluid shows a permanent agitation (the so-called Brownian motion). Another example is given by electrical resistors which always show spontaneous voltage fluctuations (the so-called Nyquist noise). These molecular processes and fluctuating forces are neither macroscopic nor microscopic. They belong to an intermediate range where molecular disorder produces random collective motions of all scales. We shall use the name "thermal agitation" in connection with this kind of motion and attempt to clarify the relation with turbulence.

It is worth noticing that electrical engineers long ago recognized the importance of thermal agitation in tubes, solid state devices, and ordinary resistors. Anyone tuning a frequency-modulation radio receiver hears a frying noise between the various stations—it is caused by thermal agitation. This noise cannot be explained by the equations of Kirchoff or Maxwell. It corresponds to the low frequency end of a very wide spectrum of electronic fluctuations. In practice, this noise limits the sensitivity of radio receivers and it also acts as initial perturbation whenever a circuit becomes electrically unstable.

An excellent example is furnished by the most sensitive receiver—the super-regenerative circuit. It consists of a circuit, say tuned at 30 megacycles and periodically allowed to become unstable, say every tenth microsecond. Each time it becomes unstable, thermal agitation triggers a burst of 30-Mc oscillation. The bursts have no phase relations since thermal noise is highly randomized. When a weak signal is received, it can act as initial perturbation and trigger several successive bursts. The bursts are then exactly synchronized. The presence or absence of the signal can be detected by comparing the phases of successive bursts, and it is obvious that to be detected, the signal must somehow rise above the thermal noise.

Since thermal agitation can influence electric systems at the rate of 10^5 decisions per second, it may also have important but yet unknown effects on fluid flows. For the purpose of collecting evidence of such interactions, we have attempted to treat the problem in the language of the continuum theory, rather than in that of statistical mechanics.

The presence of fluctuations in all dissipative systems is assured by the so-called fluctuation-dissipation theorem [Callen *et al.* (*2*, *3*)]. In order to avoid abstract concepts, we do not invoke this theorem in this report. Furthermore, it is not clear how the theorem applies to nonlinear systems in nonequilibrium.

The presence of fluctuations has a direct bearing on the question of how far the continuum theory does apply. In isotropic turbulence we found experimental evidence that the third derivative of the fluid velocity with respect to a space coordinate cannot be clearly separated from thermal agitation [Betchov (*1*)]. Since the vorticity equation contains derivatives of this order, it seems that thermal fluctuations should also contribute to this equation.

After this description of our objectives and method of approach, we must warn the reader that he shall encounter several heuristic assumptions. Clearly, the conclusions will rest on the validity of these assumptions.

II. Viscous and Incompressible Fluids

A. Brownian Motion and the Equations of Navier-Stokes

If a small particle is floating in a fluid, in the absence of external forces, it is well known that it presents an agitated motion. The averaged energy, per degree of freedom of the particle is $\frac{1}{2}kT$. This is called the Brownian motion, and its molecular origin has been studied in detail (Einstein, Perrin, Uhlenbeck, etc.)

On the other hand, we consider that the motion of viscous fluids is governed by the equations of Navier and Stokes where the pressure and the viscous stresses are the only recognized effects of molecular agitation. These equations, however, cannot predict the Brownian motion. In the absence of external forces, they always lead to complete dissipation of the energy. Therefore a study of fluid flow and of thermal agitation must begin by a modification of the basic equations. We shall confine our attention to incompressible flows and write the following equations (with summation over repeated indices):

$$\frac{\partial u_j}{\partial t} + u_k \frac{\partial u_j}{\partial x_k} - \nu \nabla^2 u_j + \frac{1}{\rho}\frac{\partial p}{\partial x_j} = n_j \tag{2.1}$$

$$\frac{\partial u_j}{\partial x_j} = 0 \tag{2.2}$$

where n_j is a random vector that we shall name the kinematic noise. Since we are not interested in acoustical fluctuations, we shall assume

$$\frac{\partial n_j}{\partial x_j} = 0 \tag{2.3}$$

It is now evident that n_j will always maintain a certain agitation in the fluid. In a closed vessel, after the initial eddies have been dissipated we can use the following linearized equations:

$$\frac{\partial u_j}{\partial t} - \nu \nabla^2 u_j = n_j \tag{2.4}$$

where we have dropped the pressure term since $\nabla^2 p = 0$.

B. The Spectrum of the Kinetic Noise

In order to determine the spectrum of n_j we must consider the equipartition of energy. In a system having a finite number of degrees of freedom, this task is simple. For a fluid, we must somehow consider an infinity of degrees of freedom, and we shall proceed as follows. We define the "measurable" velocity v_j as the average of u_j over a box of dimensions $L_1 L_2 L_3$, that is

$$v_j = (L_1 L_2 L_3)^{-1} \int_{-L_{1/2}}^{L_{1/2}} \int_{-L_{2/2}}^{L_{2/2}} \int_{-L_{3/2}}^{L_{3/2}} u_j \, dx^3 \tag{2.5}$$

We can regard the quantities L_j as the dimensions of an anemometer or a suspended particle used to trace the flow. We define the correlation tensor of u_j as

$$H_{kl}(\xi, \tau \mid u) = \langle u_k(x + \xi, t + \tau) \, u_l(x, t) \rangle \tag{2.6}$$

where the brackets indicate an ensemble average or any acceptable substitute. We define the spectral tensor of u_j as

$$\phi_{kl}(\alpha, \omega \mid u) = \frac{1}{(2\pi)^4} \int H_{kl}(\xi, \tau \mid u) \, e^{i\alpha\xi + i\omega\tau} \, d\xi^3 \, d\tau \tag{2.7}$$

where the limits of integration, when not shown, are always infinity.

It is also helpful to define ordinary Fourier transforms by assuming that u_j vanishes outside of a large vessel of dimensions $X_1 X_2 X_3$ and a large time interval D.

This leads to

$$\varphi_j(\alpha, \omega \mid u) = \frac{1}{2\pi^4} \int u_j(x, t) \, e^{-i\alpha x - i\omega t} \, dx^3 \, dt \tag{2.8}$$

with the relation

$$\varphi^*(\alpha, \omega \mid u) = \varphi(-\alpha, -\omega \mid u) \tag{2.9}$$

The following equation can be demonstrated

$$\langle \varphi_k(\alpha, \omega \mid u) \, \varphi_l(\beta, \kappa \mid u) \rangle = \frac{X_1 X_2 X_3 \, D}{(2\pi)^4} \phi_{kl}(\alpha, \omega \mid u) \, \delta(\alpha + \beta) \, \delta(\omega + \kappa) \tag{2.10}$$

The Fourier transform of v_j, from Eqs. (2.5) and (2.8) now becomes

$$\varphi_j(\alpha, \omega \mid v) = \varphi_j(\alpha, \omega \mid u) \frac{\sin \alpha_1 L_1/2}{\alpha_1 L_1/2} \frac{\sin \alpha_2 L_2/2}{\alpha_2 L_2/2} \frac{\sin \alpha_3 L_3/2}{\alpha_3 L_3/2} \tag{2.11}$$

Using some condensed notation, we use Eq. (2.10) and write

$$\phi_{kl}(\alpha, \omega \mid v) = \phi_{kl}(\alpha, \omega \mid u) \left(\frac{\sin \alpha L/2}{\alpha L/2}\right)^{\text{"6"}} \tag{2.12}$$

This simply means that the averaging process truncates the spectrum near $\alpha L = \pm\pi$. From the flow equations (2.4), we derive a relation between the spectra of u_j and of n_j:

$$\phi_{kl}(\alpha, \omega \mid u) = \frac{\phi_{kl}(\alpha, \omega \mid n)}{\nu^2\alpha^4 + \omega^2} \tag{2.13}$$

We are now ready to introduce the assumption of equipartition of energy. We shall now specify that, for any size of the box $L_1L_2L_3$ we have

$$\tfrac{1}{2}\, \rho\, L_1L_2L_3 \langle v_j v_j \rangle = \tfrac{3}{2}\, kT \tag{2.14}$$

This equation can be justified by considering the average velocity of N molecules moving at random with a mean kinetic energy $\frac{1}{2}kT$. The square of this averaged velocity is proportional to N^{-1}, and therefore the kinetic energy of the averaged motion is simply $\frac{1}{2}kT$. This corresponds to the equation

$$\tfrac{1}{2}\, \rho \int \frac{\phi_{kk}(\alpha, \omega \mid n)}{\nu^2\alpha^4 + \omega^2} \left(\frac{\sin \alpha L/2}{\alpha L/2}\right)^{\text{"6"}} d(\alpha L)^3\, d\omega = \tfrac{3}{2}\, kT \tag{2.15}$$

By virtue of the equation

$$\int \left(\frac{\sin x}{x}\right)^2 dx = \pi \tag{2.16}$$

and by assuming that Eq. (2.15) applies to any box, one finds:

$$\int \frac{\phi_{kk}(\alpha, \omega \mid n)}{\nu^2\alpha^4 + \omega^2}\, d\omega = \frac{3}{(2\pi)^3} \cdot \frac{kT}{\rho} \tag{2.17}$$

It is now appropriate to assume that the spectrum of n_j is isotropic, and we write:

$$\phi_{kl} = \nu\alpha^2\, F(\alpha, \omega) \left[\delta_{kl} - \frac{\alpha_k\, \alpha_l}{\alpha^2}\right] \tag{2.18}$$

This leads from Eq. (2.17) to

$$\int \frac{F(\alpha, \omega)}{1 + (\omega/\nu\alpha^2)^2}\, d\left(\frac{\omega}{\nu\alpha^2}\right) = \frac{3}{16\pi^3} \frac{kT}{\rho} \tag{2.19}$$

This relation cannot hold unless F is a function of $\omega/\nu\alpha^2$ alone. We shall now assume that F is constant. This is equivalent to saying that, for sufficiently low frequencies, the spectrum of n_j is white. If we suppose that n_j is the sum of brief uncorrelated impulses, it follows that F is independent of ω. Therefore, it appears that we must recognize something else about n_j besides the equipartition of energy. This difficulty can be avoided by using directly the fluctuation-dissipation theorem [Callen *et al.* (*2, 3*), also Rytov (*6*)].

The integration of Eq. (2.19) is now immediate, and it leads to the spectrum of kinetic noise

$$\phi_{kl}(\alpha, \omega \mid n) = \frac{3}{(2\pi)^4} \nu\alpha^2 \frac{kT}{\rho} \left[\delta_{kl} - \frac{\alpha_k \alpha_l}{\alpha^2}\right] \tag{2.20}$$

C. Kinetic Fluctuations

Some insight on the fluctuations can be gained by defining another vector by the relation

$$n_j = \nu \nabla^2 w_j \tag{2.21}$$

The spectrum of w_j is given by

$$\phi_{kl}(\alpha, \omega \mid w) = \frac{3}{(2\pi)^4} \alpha^{-2} \frac{kT}{\rho\nu} \left[\delta_{kl} - \frac{\alpha_k \alpha_l}{\alpha^2}\right] \tag{2.22}$$

As long as the inertial forces are unimportant it follows from Eq. (2.4) that $u_j = w_j$. When the frequency becomes comparable with $\nu\alpha^2$, the fluid no longer responds to the noise w_j, as shown by the following equation obtained from Eqs. (2.13) and (2.21):

$$\phi_{kl}(\alpha, \omega \mid u) = \frac{\phi_{kl}(\alpha, \omega \mid w)}{1 + (\omega/\nu\alpha^2)^2} \tag{2.23}$$

Thus, the inertia cuts off the response of the fluid to thermal collisions with a band width of about $\omega = \pm \pi/2\nu\alpha^2$. If we now average over a box L^3 we find that it introduces a wave number cutoff near $\alpha L = \pi$. We therefore can write

$$\langle v_j v_j \rangle \approx \int_{-\pi/L}^{+\pi/L} d\alpha^3 \int_{-\pi/2\nu\alpha^2}^{+\pi/2\nu\alpha^2} \phi_{kk}(\alpha, \omega \mid w)\, d\omega \approx 3\frac{kT}{\rho L^3} \tag{2.24}$$

Thus w_j indicates the velocity fluctuations as long as inertia and space average are unimportant. With a "viscous" frequency cutoff, it leads simply to our basic assumption of equipartition.

The vorticity of the fluid can be introduced and for isotropic motion one has simply

$$\phi_{kl}(\alpha, \omega \mid \text{curl } u) = \frac{3}{(2\pi)^4} \frac{kT}{\rho\nu} \frac{[\delta_{kl} - (\alpha_k \alpha_l/\alpha^2)]}{1 + (\omega/\nu\alpha^2)^2} \tag{2.25}$$

This means that the spectrum of the vorticity fluctuations is constant as long as inertia is negligible. It also signifies that there is no such thing as an irrotational flow.

It is also easy to show that the averaged vorticity of a box corresponds to a rotational random motion of the fluid element, with energy $\frac{1}{2}kT$ about any axis of rotation.

D. Fluctuations in a Shear Flow

In a shear flow the fluid is no longer in thermodynamic equilibrium, since mechanical energy is constantly transformed into heat. If we separate the fluid velocity into a mean motion $U_j(x)$ and a small fluctuation $u_j(x, t)$, the linearized equations of motion are

$$\frac{\partial u_j}{\partial t} - \nu \nabla^2 u_j + \left[U_k \frac{\partial u_j}{\partial x_k} + \frac{\partial U_j}{\partial x_k} u_k - \frac{1}{\rho} \frac{\partial p}{\partial x_j}\right] = n_j \tag{2.26}$$

$$\frac{1}{\rho} \nabla^2 p = -2 \frac{\partial U_k}{\partial x_j} \frac{\partial u_j}{\partial x_k} \tag{2.27}$$

The bracket indicates the terms due to coupling with the mean flow. It is tempting to assume that n_j is the same as previously, but it is by no means certain that the presence of shear leaves n_j unchanged. Let us consider a Couette flow along the x axis, with $dU/dy = G$. A small fluid particle at the origin is subject to shear stresses σ_{xy} because every exchange of molecules across a face $\Delta x \Delta z$ contributes a momentum of the order of $G\lambda$ where λ is the mean free path. Since G is constant, there is no net force on the particle. However, these exchanges also produce a net torque on the particle, and somehow the stresses σ_{yx} appear and exactly balance this torque. This presence of stresses σ_{yx} cannot be explained unless the molecular velocity distribution is not exactly Maxwellian. This implies a modification of n_j, which could affect primarily the spectral components of low frequency and low wave number. For lack of better knowledge, we shall assume that n_j is the same as in the condition of thermodynamical equilibrium.

E. Dynamic Response of a Shear Flow

If n_j is known, the fluctuations can be determined, at least in a statistical sense. The first step would be to solve the homogeneous form of Eq. (2.26), that is, for $n_j = 0$. It is well known that these equations sometimes are unstable. Thus, a laminar boundary layer is stable as long as the Reynolds number $\mathrm{Re} = U_0\delta^*/\nu$ is less than some critical value, say $\mathrm{Re}_c = 700$. Beyond a certain distance from the leading edge, the flow is unstable and the perturbations grow exponentially. It is therefore essential to understand how the noise will trigger these unstable modes of motion. The study of simpler systems suggests that, when the flow changes from stable to unstable condition, it gathers the effects of the forcing function during a transit time T_c. This is discussed in an appendix (see Section IV). Once the flow has become unstable, the perturbations grow as long as the linearized equations (2.26) are valid. There is some evidence that the growth of large perturbations is no longer exponential, but another phenomenon soon appears: transition to turbulent flow.

The points of greatest interest to us are that a shear flow somehow becomes very sensitive to small perturbations and that they may have cumulative effects.

An essential property of a laminar shear flow is that the coupling with the mean flow tends to oppose the inertia effects. Energy is fed from the mean flow to the perturbations. Thus, in place of a frequency cutoff determined by the viscosity, we can expect a cutoff near some higher frequency, such as U_0/δ^* where U_0 is a free-stream velocity and δ^* a characteristic length. Broadly speaking, a shear flow is more ready to respond to fluctuating forces than a fluid at rest or a fluid in uniform motion. Another essential feature of boundary layer flows is that the viscous stresses are generally small but all important because they control the rate of growth or decay of the perturbations. This does not apply to jets and wakes where the inviscid flow is unstable.

In the absence of more precise ways of predicting the response of shear flows to fluctuating forces, we shall make the assumption that the "viscous" band-width $\nu\alpha^2$ must be replaced by a broader "inertial" bandwidth of the order of U_0/δ^*.

We shall not take the transit time T_c into consideration on the ground that it has the same effects as a broader bandwidth. Thus our "inertial" bandwidth includes also the effects of transit processes.

By analogy with Eq. (2.24), we can now estimate the fluctuations in a box of dimension δ^* with an "inertial" bandwidth:

$$\langle v_j v_j \rangle \approx \int_{-\pi/\delta^*}^{\pi/\delta^*} d\alpha^3 \int_{-\pi U_0/2\delta^*}^{\pi U_0/2\delta^*} \phi_{kk}(\alpha, \omega \mid w)\, d\omega \approx \frac{3kT}{\rho}\, \delta^{*-3}\, \frac{U_0\delta^*}{\nu} \qquad (2.28)$$

The fluid simply responds to a broader spectrum of the forcing function than in the absence of shear.

F. Transition in a Boundary Layer

Let us consider a simple perturbation in a laminar boundary layer along a flat plate, from the point x_c where the motion becomes unstable to some location x. If $\beta(x)$ gives the rate of growth of a perturbation, we have

$$\langle u^2 \rangle = u_0^2 \exp 2 \int_{x_c}^{x} \beta \, dx \tag{2.29}$$

where u_0 plays the role of an initial perturbation. The integral $\int \beta \, dx$ has been evaluated by A. M. O. Smith *et al.* (*8*, *9*) between x_c and the point at which turbulence appears. He studied a variety of flows and found that it amounts to e^9, that is, a factor of about $10^{3.5}$. This permits us to evaluate the magnitude of the perturbations caused by thermal agitation, just prior to turbulent transition. In air, at normal conditions, we find in cgs units:

$$\frac{\langle u^2 \rangle}{U_0^2} = \frac{2 \times 10^{-3}}{U_0 \delta^{*2}} \tag{2.30}$$

With $U_0 = 15$ m/sec and Re $= 700$, this leads to $\delta^* = 0.07$ cm and preturbulent fluctuations of 1.7%. This level of velocity fluctuations is large enough to suggest that thermal agitation can cause turbulence, when the other causes of disorder have been sufficiently reduced. This would explain the relative constance of the factor e^9.

The preturbulence fluctuations would be larger if we had assumed that the sensitive portion of the layer is smaller than δ^*. In this respect the role of the critical layer should be scrutinized.

The assumption of an enlarged bandwidth can be abandoned and replaced by the effects of the transit time T_c. From the results of Shen (*7*), reproduced in Lin (*5*), we have found for $\alpha\delta^* = 0.2$ that at the beginning of instability, $d\beta/d\mathrm{Re}$ is of the order of $2 \times 10^{-5} U_0/\delta^*$. From the relation

$$\delta^* = 1.7 \left(\frac{\nu x}{U_0} \right)^{1/2} \tag{2.31}$$

one arrives at:

$$\frac{d\,\mathrm{Re}}{dx} = \frac{0.85}{\delta^*} \tag{2.32}$$

and from Eqs. (2.4-2.8) one finds:

$$\omega T_c \approx 400 \tag{2.33}$$

This leads to initial perturbations of about the same size than with the assumption of inertial bandwidth. If transit and inertial bandwidth are both contributing, the preturbulence perturbations are of the order of 30% of the free stream velocity, a very large figure.

G. Transition in a Jet

Let us consider a jet of air of diameter $d = 0.5$ cm and velocity $U_0 = 30$ cm/sec as could be produced by a burning cigarette. The Reynolds number is about Re $= 100$ and a value of about 4 is sufficient for unstable behavior. Taking d^3 as the sensitive volume and U_0/d as the cutoff frequency (or some effect of transit), we find for the initial velocity fluctuations

$$u_0^2 = \frac{kT}{\rho} d^{-3} \frac{Ud}{\nu} = 2.4 \times 10^{-8} \text{ cm}^2 \text{ sec}^{-2} \tag{2.34}$$

The rate of growth is much larger than in boundary layers because of the absence of solid walls. It can be estimated at $\beta = 0.2d^{-1}$.

The amplification necessary to produce fluctuations of 1% of the free-stream velocity is about $e^{8.5}$, and this is reached at a distance of about 20 cm. This suggests that thermal agitation could be the cause of very familiar forms of turbulence.

H. Transition in a Turbulent Flow

Let us consider homogeneous, isotropic, and incompressible turbulence with eddies of size $d = 2$ cm and eddy velocities of the order of $U_0 = 5$ cm/sec. This corresponds to a rather low eddy Reynolds number of the order of 70. Locally, the flow will produce highly unstable three-dimensional shear flows, and we can expect amplification of the thermal motion. If the initial perturbations are given by a relation such as

$$u_0^2 = \frac{kT}{\rho} d^{-3} \frac{U_0 d}{\nu} \tag{2.35}$$

it is easy to find that an amplification of e^{12} will produce new eddies as large as the original eddies. If the rate of growth is of the order of d^{-1}, this requires such a long time that the turbulence would decay before the amplification is performed. However, if the big eddies are sustained by some external process, such as a gross instability, we can expect a steady supply of completely random fluctuations, emerging out of the kinetic noise w_j.

For larger eddy Reynolds numbers, the amplification required for a similar effect is much larger but since it may be furnished in several stages, the basic idea can still be used. Thus, the cascade of energy from large eddies to small eddies is perhaps associated with the constant build up of unpredictable new eddies, out of the thermal agitation.

It has been observed by D. Coles (*4*) and independently by J. Laufer (private communication) that turbulence can stop as rapidly as it can appear. Such behavior supports the view that thermal motion may have something to contribute to the theory. Indeed if the amplification mechanism is turned off, the random motion will stop just as rapidly as it appears when the amplification is applied. The forcing functions n_j can be regarded as a time-reversible phenomena, and it is only the viscous stresses which bear the mark of irreversibility.

III. Incompressible and Conductive Fluids

A. Electric Noise

It is well known that every fluid or solid conductor of electricity is also the source of spontaneous voltage fluctuations. From the point of view of magnetohydrodynamics, this means that the magnetic field h_j fluctuates constantly and that the equipartition of energy must be extended to the magnetic energy density $\frac{1}{2}\mu h_j h_j$. In magnetohydrodynamics, we use the following incompressible equations,

$$\frac{\partial h_j}{\partial t} + u_k \frac{\partial h_j}{\partial x_k} - h_k \frac{\partial u_j}{\partial x_k} - \frac{1}{\mu\sigma} \nabla^2 h_j = m_j \tag{3.1}$$

$$\frac{\partial h_j}{\partial x_j} = 0 \tag{3.2}$$

where we have added a new term m_j that we shall call the magnetic noise.

In a fluid at rest, these equations reduce to diffusion equations, and the procedure to determine the spectrum of m_j is exactly the same as for the kinetic noise, since the equations have exactly the same form. We therefore write directly:

$$\phi_{kl}(\alpha, \omega \mid m) = \frac{3}{(2\pi)^4} \frac{\alpha^2}{\mu\sigma} \frac{kT}{\mu} \left[\delta_{kl} - \frac{\alpha_k \alpha_l}{\alpha^2}\right] \tag{3.3}$$

Indeed, the roles of h_j, μ, and $1/\mu\sigma$ are equivalent to those of u_j, ρ, and ν.

By proper use of the relations

$$\text{curl } h = J \tag{3.4}$$

$$J_j = \sigma E_j \tag{3.5}$$

one can obtain the spectrum of the electric field:

$$\phi_{kl}(\alpha, \omega \mid E) = \frac{3}{(2\pi)^4} \frac{kT}{\sigma} \frac{[\delta_{kl} - (\alpha_k \alpha_l/\alpha^2)]}{1 + \mu\sigma\omega^2/\alpha^4} \tag{3.6}$$

B. The Nyquist Noise

It is not too difficult to show that these fluctuations of the electric field give exactly the Nyquist noise. Considering a conductor of size $L_1L_2L_3$, the E_1 component must be averaged over a cross section L_2L_3 and integrated along the center line. This gives the potential difference V as:

$$V = \int_0^{L_1} \int_0^{L_2} \int_0^{L_3} E_1 \frac{dx^3}{L_2L_3} \tag{3.7}$$

It is necessary to assume that we are interested only in the low frequency components of E_1, say in the range $|\omega| \leq 2\pi f$. One is then led to the result

$$\langle V^2 \rangle = \frac{4L_1}{L_2L_3} \frac{kT}{\sigma} f \tag{3.8}$$

In the developments, the following relation is useful:

$$\int F(\alpha) \left[1 - \frac{\alpha_1^2}{\alpha^2}\right] d\alpha^3 = \frac{2}{3} \int F(\alpha)\, d\alpha^3 \tag{3.9}$$

Since $L_1/\sigma L_2L_3$ is simply the ohmic resistance R_σ of the sample, Eq. (3.8) corresponds to the Nyquist formula:

$$\langle V^2 \rangle = 4\, R_\sigma\, kT\, f \tag{3.10}$$

If the voltage fluctuations are observed with an ideal instrument and with a cubic sample of material, the conductive cutoff must be considered and one obtains

$$\langle V^2 \rangle = \frac{8\pi^2}{L^3} \frac{kT}{(\mu\sigma^3)^{1/2}} \tag{3.11}$$

By using large samples of good conductors and instruments responding up to 10 Mc, one should be able to observe this effect of the frequency cutoff.

C. Fluctuations in a Sheared Plasma Flow

In a plasma flow, and with all the limitations inherent in the MHD approach, we can expect the presence of kinetic and magnetic noise, and the occurrence of instabilities, involving both the kinematic and the magnetic modes. This creates a problem of great difficulty. Let us try to make a crude guess, in the case of fluctuations corresponding to a sensitive volume of 1 cc, at a temperature of 10^5 Kelvin, with an amplification in amplitude of 10^4. We use a formula such as

$$h_0^2 = \frac{kT}{\mu} d^{-3} K \tag{3.12}$$

where K is a magnetic Reynolds number or some kind of transit time. With K of the order of 10^3, we find magnetic fluctuations of the order of 3 gauss. Assuming a proton density of the order of 10^{16} per cc, the same example leads to velocity fluctuations of the order of 30 cm/sec.

IV. Variable Damping and Transit Time

The response of a system of variable damping to random impulses can be examined by reference to a simple model. We are particularly interested in the behavior of the system as the damping changes from positive to negative values. This initiates a divergent solution, with an initial amplitude determined by the random impulses.

Let us consider an oscillator where the dependent variable f is a function of the time t satisfying the equation

$$\omega^{-2} \frac{d^2 f}{dt^2} + 2\,\omega^{-1} k(t) \frac{df}{dt} + f = n(t) \tag{4.1}$$

where ω is constant, k is a given function of t, and n is a random function consisting of uncorrelated positive and negative pulses. There are N pulses per second, each with a short duration τ. The time integral over a single pulse gives its strength a. For $\omega\tau < 1$, the spectrum of n is constant and equal to

$$\phi\,(\omega \mid n) = Na^2 \tag{4.2}$$

The damping k is a slowly varying function of t. The solution corresponding to a single impulse occurring at time $t = s$ is therefore

$$f = 0 \qquad \text{if} \quad t < s$$

$$f = a\omega \exp\left[-\,\omega \int_s^t k(r)\,dr\right] \sin \omega(t - s) \qquad \text{if} \quad t > s \tag{4.3}$$

This assumes that f and df/dt start from zero at $t = -\infty$, and it also assumes that k^2 is less than unity. The responses to single impulses can be superposed to form the response to the noise since the equation is linear. Because the impulses are uncorrelated, their effect on the amplitude of f will add quadratically. With the angular brackets indicating an ensemble average or some acceptable substitute, we have therefore

$$\langle f(t)^2 \rangle = Na^2\omega^2 \int_{-\infty}^{t} \exp\left[-2\omega \int_{s}^{t} k(r)\, dr\right] ds \tag{4.4}$$

If k varies from some positive value for $t \ll 0$ to some negative value for $t \gg 0$ with a zero at $t = 0$, the largest contribution to the integral of Eq. (4.4) comes from the vicinity of $t = 0$. We can define the time T by the following relation

$$T = \int_{-\infty}^{t} \exp\left[2\omega \int_{0}^{s} k(r)\, dr\right] ds \tag{4.5}$$

If t is sufficiently large, T approaches a limit T_c independent of t, since the integral in r becomes very negative. We also define the "equivalent" single perturbation

$$f_0^2 = 2\phi\,(\omega \mid n)\, \omega^2\, T_c \tag{4.6}$$

We can now rewrite Eq. (4.4) as

$$\langle f(t)^2 \rangle = \tfrac{1}{2} f_0^2 \exp\left[-2\omega \int_{0}^{t} k(r)\, dr\right] \tag{4.7}$$

It indicates that the amplitude at a time such that k is already sufficiently negative can be viewed as the result of a single pulse at $t = 0$, with a factor 1/2 coming from the mean square of the sine wave encountered in Eq. (4.3). It is not surprising to find f_0^2 proportional to $\phi(\omega \mid n)$ but the role of T_c needs further clarification.

Let us assume that k is very large for $t < 0$, that k is zero for $0 < t < T_1$, and that $k = -1$ for $t > 0$. The pulses occurring prior to $t = 0$ have no effect on $f(t)$. Each pulse occurring when k is zero simply switches on a sine wave of frequency ω. At $t = T_1$, the solution is a sum of NT_1 sine waves of random phases and equal amplitudes. From this time on, the amplitude will grow exponentially, with new contributions as new pulses occur. In the interval $0 < t < T_1$ the solution accumulates the effects of single pulses and the real explosive growth does not begin until time $t = T_1$.

From Eq. (4.5) we find $T_c \approx T_1$, and all the general features of the model are easily identified in Eq. (4.6). It can also be shown that Eq. (4.6) is valid if $\phi(\omega \mid n)$ is a slowly varying function of ω.

It therefore seems appropriate to call T_c the transit time, since it represents the time during which the system, at zero damping, would collect the appropriate amount of "energy." We also remark that the integral occurring in Eq. (4.7) plays exactly the role of the amplification factor studied by A. M. O. Smith.

Finally, a useful formula can be obtained by considering the case where dk/dt is almost constant. From Eq. (4.5) one finds

$$T_c = \left(\frac{2\pi}{dk/dt}\right)^{1/2} \tag{4.8}$$

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