Homework No.6, 553.483/683, Due March 15, 2024.

Problem 1. Use difference calculus to answer each of the following questions:
(a) Strikwerda, 1st Ed., Exercise 3.2.2 (2nd Ed., Exercise 3.3.2)
(b) Strikwerda, 1st Ed., Exercise 3.2.3 (2nd Ed., Exercise 3.3.3)

Problem 2. Use difference calculus to show that the following improved Crank-Nicolson scheme for $v_{t} + av_{x} = f$

$$\frac{1}{6}v_{m+1}^{n+1} + \frac{2}{3}v_{m}^{n+1} + \frac{1}{6}v_{m-1}^{n+1} + \frac{a\lambda}{4}(v_{m+1}^{n+1} - v_{m-1}^{n+1}) = \frac{1}{6}v_{m+1}^{n} + \frac{2}{3}v_{m}^{n} + \frac{1}{6}v_{m-1}^{n} - \frac{a\lambda}{4}(v_{m+1}^{n} - v_{m-1}^{n})$$

$$+ \frac{k}{12}(f_{m+1}^{n+1} + 4f_{m}^{n+1} + f_{m-1}^{n+1} + f_{m+1}^{n} + 4f_{m}^{n} + f_{m-1}^{n})$$

is accurate of order $(2, 4)$. Show also that the scheme is unconditionally stable.

Problem 3. Strikwerda, 1st Ed., Exercise 3.3.1 (2nd Ed., Exercise 3.4.1). For specificity, take $a = 1$ and then solve with initial data $u_{0}(x) = \exp(-49(x - 0.1)^{2})$ over the time interval $0 \leq t \leq 1$. For each method take $h = 1/200$ and plot the numerical solutions together with the exact solution at each of the times $t = 0.25, 0.5, 1$. Calculate also the relative error in the $L^{2}$-norm at the final time $t = 1$.

Problem 4. (a) Strikwerda, 1st Ed., Exercise 3.4.3 (2nd Ed., Exercise 3.5.8)
(b) Using the algorithm from part (a), write a code thomasper2 to calculate the solution of a general tridiagonal system with periodic boundary conditions. For simplicity, you may assume that $\text{sign}(a_{1}b_{1}) > 0$ and you may, if you wish, use as a subroutine the course code thomasNN for Neumann boundary conditions. Verify numerically that your code gives the same result as the course code thomasper.

Problem 5. (a) Write a code CrankNicholsonExp to implement the Crank-Nicholson scheme for the wave equation $u_{t} + au_{x} = 0$ with $a > 0$ using the explicit, second-order conditions at the right boundary:

$$v_{M}^{n+1} = 2v_{M-1}^{n} - v_{M-2}^{n}$$

which corresponds to quasi-characteristic extrapolation.
(b) Use your code to approximate the initial-boundary-value in Problem 3 and compare with the result from CrankNicholson implementing the second-order implicit b.c. in the following two cases: (i) $h = 1/100$, $\lambda = 0.8$ and (ii) $h = 1/250$, $\lambda = 4$. Plot the numerical solutions together with the exact solution at each of the times $t = 0.25, 0.5, 1$ and calculate also the relative error in the $L^{2}$-norm at the final time $t = 1$. When is the implicit boundary condition superior to the explicit one?