Problem 1. For each of the following problems, prove consistency of the stated scheme for the wave equation with
\[ P = \partial_t + a \partial_x \] and give the order of \( Pu - P_{k,h} u \) in terms of \( k \) and \( h \).
(a) Strickwerda 1.4.1.
(b) Strikwerda 1.4.2.
(c) Strickwerds 1.4.3.

(a) Expanding in a series around \( t \) and \( x \) respectively we get
\[
u(t + k, x) = u(t, x) + k \partial_t u(t, x) + \frac{1}{2} k^2 \partial^2_t u(t, x) + O(k^3),
\]
and
\[
u(t, x + h) = u(t, x) + h \partial_x u(t, x) + \frac{1}{2} h^2 \partial^2_x u(t, x) + \frac{1}{6} h^3 u(t, x) + O(h^4).
\]
We find that
\[
u(t + k, x) - \frac{\nu(t, x)}{k} = \partial_t u(t, x) + O(k), \quad \frac{\nu(t, x + h) - \nu(t, x - h)}{2h} = \partial_x u(t, x) + O(h^2).
\]
Thus
\[(P - P_{k,h})u = O(k) + O(h^2).\]

(b) Expanding in a series around \( t \) we get
\[
u(t \pm k, x) = u(t, x) \pm k \partial_t u(t, x) + \frac{1}{2} k^2 \partial^2_t u(t, x) \pm \frac{k^3}{6} \partial^3_t u(t, x) + O(k^4).
\]
The expansion in \( x \) is identical to in (a). Thus
\[
u(t + k, x) - \frac{\nu(t - k, x)}{2k} = \partial_t u(t, x) + O(k^2),
\]
and
\[(P - P_{k,h})u = O(k^2) + O(h^2).\]

(c) Expanding in a series around \( (t, x) \) we get
\[
u(t + k, x + h) = u(t, x) + k \partial_t u(t, x) + h \partial_x u(t, x) + \frac{1}{2} k^2 \partial^2_t u(t, x) + \frac{1}{2} h^2 \partial^2_x u(t, x) + kh \partial_t \partial_x u(t, x) + \cdots.
\]
Then
\[
u(t + k, x + h) - \frac{\nu(t + k, x)}{h} = \partial_x u(t, x) + O(k) + O(h).
\]
We use our expansions in (a) and (b) to write
\[ P_{k,h} u = \partial_t u + O(k) + a \left[ \frac{\partial_x u + O(k) + O(h)}{2} + \frac{\partial_x u + O(h)}{2} \right], \]
and since \( Pu = f \)
\[ P_{k,h} u - f = (P_{k,h} - P)u = O(k) + O(h). \]
First note that

\[ 2|v_{m+1}^n| |v_{m-1}^n| \leq |v_{m+1}^n|^2 + |v_{m-1}^n|^2, \]

so we can write

\[ |v_{m+1}^{n+1}|^2 = |\alpha|^2 |v_{m+1}^n|^2 + 2|\alpha\beta||v_{m+1}^n||v_{m-1}^n| + |\beta|^2 |v_{m-1}^n|^2 \]
\[ \leq (|\alpha|^2 + |\alpha||\beta|) |v_{m+1}^n|^2 + (|\beta|^2 + |\alpha||\beta|) |v_{m-1}^n|^2. \]

Thus

\[ \sum_{m \in \mathbb{Z}} |v_{m+1}^{n+1}|^2 \leq (|\alpha|^2 + 2|\alpha| + |\beta|^2) \sum_{m \in \mathbb{Z}} |v_m^0|^2 = (|\alpha| + |\beta|)^2 \sum_{m \in \mathbb{Z}} |v_m^0|^2. \]

If we iterate \( n = 0, 1, 2, \ldots \) we find that

\[ \sum_{m \in \mathbb{Z}} |v_m^n|^2 \leq (|\alpha| + |\beta|)^{2n} \sum_{m \in \mathbb{Z}} |v_m^0|^2. \]

Thus, such schemes are stable when \( |\alpha| + |\beta| \leq 1 \). In Lax–Friedrichs we have \( \alpha = (1 + a\lambda)/2 \) and \( \beta = (1 - a\lambda)/2 \). If \( a\lambda < 1 \) we have \( \alpha > 0, \beta > 0 \) and thus

\[ |\alpha| + |\beta| = \alpha + \beta = 1. \]

This gives stability.
After multiplying we obtain
\[ |v^{n+1}_m|^2 - |v^n_m|^2 + a\lambda(v^{n+1}_m + v^n_m)(v^{n+1}_m - v^n_m) = 0. \]
Thus
\[ \sum_{m \in \mathbb{Z}} |v^{n+1}_m|^2 + a\lambda \sum_{m \in \mathbb{Z}} (v^{n+1}_m v^{n+1}_{m+1} + v^n_m v^n_{m+1} - v^{n+1}_m v^n_{m-1} - v^n_m v^{n+1}_{m-1}) = \sum_{m \in \mathbb{Z}} |v^{n-1}_m|^2. \]
We can shift the sum index on the last two products of the big summand:
\[ \sum_{m \in \mathbb{Z}} |v^{n+1}_m|^2 + a\lambda \sum_{m \in \mathbb{Z}} (v^{n+1}_m v^{n+1}_{m+1} + v^n_m v^n_{m+1} - v^{n+1}_m v^n_{m-1} - v^n_m v^{n+1}_{m-1}) = \sum_{m \in \mathbb{Z}} |v^{n-1}_m|^2. \]
Rearranging the terms of the summand across the equality and adding \(|v^n_m|^2\) to both sides we obtain the desired relation:
\[ \sum_{m \in \mathbb{Z}} \left[ |v^{n+1}_m|^2 + |v^n_m|^2 + a\lambda(v^{n+1}_m v^{n+1}_{m+1} - v^{n+1}_m v^n_{m-1}) \right] = \sum_{m \in \mathbb{Z}} \left[ |v^n_m|^2 + |v^{n-1}_m|^2 + a\lambda(v^n_m v^n_{m+1} - v^{n-1}_m v^n_{m+1}) \right]. \]
Now using this relation we can "step down" the increments \(n\) and get
\[ \sum_{m \in \mathbb{Z}} \left[ |v^{n+1}_m|^2 + |v^n_m|^2 + a\lambda(v^{n+1}_m v^{n+1}_{m+1} - v^{n+1}_m v^n_{m-1}) \right] = \sum_{m \in \mathbb{Z}} \left[ |v^1_m|^2 + |v^0_m|^2 + a\lambda(v^1_m v^0_{m+1} - v^1_m v^0_{m-1}) \right]. \]
As we generally have \(2pq \geq -(p^2 + q^2)\) for any \(p, q\) we can write for all \(n\):
\[ \sum_{m \in \mathbb{Z}} (v^{n+1}_m v^{n+1}_{m+1} - v^{n+1}_m v^n_{m-1}) \geq \frac{1}{2} \sum_{m \in \mathbb{Z}} \left( |v^{n+1}_m|^2 + |v^n_m|^2 + |v^{n+1}_m|^2 + |v^n_m|^2 \right) \]
\[ = - \sum_{m \in \mathbb{Z}} \left( |v^{n+1}_m|^2 + |v^n_m|^2 \right). \]
The second step is attained by shifting the index \(m\) for the second and third terms of the summand on the right-hand side. Similarly we have \(2pq \leq p^2 + q^2\) for any \(p, q\), so for \(n = 0\) we will write
\[ \sum_{m \in \mathbb{Z}} (v^1_m v^0_{m+1} - v^1_m v^0_{m-1}) \leq \frac{1}{2} \sum_{m \in \mathbb{Z}} \left( |v^1_m|^2 + |v^0_m|^2 + |v^1_m|^2 + |v^0_m|^2 \right) \]
\[ = \sum_{m \in \mathbb{Z}} \left( |v^1_m|^2 + |v^0_m|^2 \right). \]
As before we shift the index \(m\) in the same way to obtain the second step. We thus have
\[ (1 - |a\lambda|) \sum_{m \in \mathbb{Z}} \left( |v^{n+1}_m|^2 + |v^n_m|^2 \right) \leq \sum_{m \in \mathbb{Z}} \left[ |v^{n+1}_m|^2 + |v^n_m|^2 + a\lambda(v^{n+1}_m v^n_{m+1} - v^{n+1}_m v^n_{m-1}) \right] \]
\[ = \sum_{m \in \mathbb{Z}} \left[ |v^1_m|^2 + |v^0_m|^2 + a\lambda(v^1_m v^0_{m+1} - v^1_m v^0_{m-1}) \right] \]
\[ \leq (1 + |a\lambda|) \sum_{m \in \mathbb{Z}} \left( |v^1_m|^2 + |v^0_m|^2 \right). \]
Because of the condition \(|a\lambda| < 1\), we can divide both sides by \(1 - |a\lambda| > 0\) and this gives stability.
Problem 4**. Prove for all $a\lambda \neq 1$ the stability of the scheme

$$\frac{v_{m+1}^n - v_m^n}{k} + \frac{a\lambda}{2} \left( \frac{v_{m+1}^{n+1} - v_m^{n+1}}{h} + \frac{v_{m+1}^n - v_m^n}{h} \right) = 0$$

for the wave equation $u_t + au_x = 0$. Use the suggestion of Strikwerda, 1.5.3 for $a\lambda < 1$ and of Strikwerda, 1.5.4 for $a\lambda > 1$.

First we consider $a\lambda < 1$. After following the hint we obtain

$$|v_{m+1}^n|^2 - |v_m^n|^2 + \frac{a\lambda}{2} (v_{m+1}^{n+1} + v_m^n) (v_{m+1}^{n+1} - v_m^n + v_m^n - v_{m-1}^n) = 0.$$

Distributing and rearranging we find

$$\left(1 - \frac{a\lambda}{2}\right)|v_{m+1}^n|^2 + \frac{a\lambda}{2} v_m^n v_{m+1} + \frac{a\lambda}{2} (v_m^{n+1} v_{m+1} - v_{m-1}^n v_m^n) = \left(1 - \frac{a\lambda}{2}\right)|v_m^n|^2 + \frac{a\lambda}{2} v_m^n v_{m-1}^n.$$

Summing over $m$:

$$\sum_{m \in \mathbb{Z}} \left[ \left(1 - \frac{a\lambda}{2}\right)|v_{m+1}^n|^2 + \frac{a\lambda}{2} v_m^n v_{m+1} + \frac{a\lambda}{2} (v_m^{n+1} v_{m+1} - v_{m-1}^n v_m^n) \right] = \sum_{m \in \mathbb{Z}} \left[ \left(1 - \frac{a\lambda}{2}\right)|v_m^n|^2 + \frac{a\lambda}{2} v_m^n v_{m-1}^n \right].$$

Now shift the indices in $m$ on the last terms on the left- and right-hand sides, one obtains the basic relation

$$\sum_{m \in \mathbb{Z}} \left[ \left(1 - \frac{a\lambda}{2}\right)|v_{m+1}^n|^2 + \frac{a\lambda}{2} v_m^n v_{m+1} + \frac{a\lambda}{2} (v_m^{n+1} v_{m+1} - v_{m-1}^n v_m^n) \right] = \sum_{m \in \mathbb{Z}} \left[ \left(1 - \frac{a\lambda}{2}\right)|v_m^n|^2 + \frac{a\lambda}{2} v_m^n v_{m-1}^n \right].$$

As $-(p^2 + q^2) \leq 2pq \leq p^2 + q^2$ for any $p, q$, we can get the following bounds for all $n$:

$$-\frac{|a\lambda|}{2} (|v_{m+1}^n|^2 + |v_{m+1}^n|^2) \leq a\lambda v_m^n v_{m+1} \leq \frac{|a\lambda|}{2} (|v_{m+1}^n|^2 + |v_{m+1}^n|^2).$$

Summing and shifting indices:

$$-\frac{|a\lambda|}{2} \sum_{m \in \mathbb{Z}} (|v_{m+1}^n|^2 + |v_{m+1}^n|^2) \leq a\lambda \sum_{m \in \mathbb{Z}} v_m^n v_{m+1} v_m^n \leq \frac{|a\lambda|}{2} \sum_{m \in \mathbb{Z}} (|v_{m+1}^n|^2 + |v_{m+1}^n|^2),$$

$$-|a\lambda| \sum_{m \in \mathbb{Z}} |v_{m+1}^n|^2 \leq a\lambda \sum_{m \in \mathbb{Z}} v_m^n v_{m+1} v_{m+1} \leq |a\lambda| \sum_{m \in \mathbb{Z}} |v_{m+1}^n|^2.$$

Thus

$$\left[ \left(1 - \frac{a\lambda}{2}\right) - \frac{|a\lambda|}{2} \right] \sum_{m \in \mathbb{Z}} |v_{m+1}^n|^2 \leq \sum_{m \in \mathbb{Z}} \left[ \left(1 - \frac{a\lambda}{2}\right) |v_m^n|^2 + \frac{a\lambda}{2} v_m^n v_{m+1} \right] = \sum_{m \in \mathbb{Z}} \left[ \left(1 - \frac{a\lambda}{2}\right) |v_m^n|^2 + \frac{a\lambda}{2} v_{m-1}^n v_m^n \right] \leq \left[ \left(1 - \frac{a\lambda}{2}\right) + \frac{|a\lambda|}{2} \right] \sum_{m \in \mathbb{Z}} |v_{m+1}^n|^2.$$
For $a\lambda \leq 0$,
\[
\left(1 - \frac{a\lambda}{2}\right) - \frac{|a\lambda|}{2} = \left(1 + \frac{|a\lambda|}{2}\right) - \frac{|a\lambda|}{2} = 1 > 0
\]
\[
\left(1 - \frac{a\lambda}{2}\right) + \frac{|a\lambda|}{2} = \left(1 + \frac{|a\lambda|}{2}\right) + \frac{|a\lambda|}{2} = 1 + |a\lambda| > 0
\]

For $0 < a\lambda < 1$
\[
\left(1 - \frac{a\lambda}{2}\right) - \frac{|a\lambda|}{2} = \left(1 - \frac{a\lambda}{2}\right) - \frac{a\lambda}{2} = 1 - a\lambda > 0
\]
\[
\left(1 - \frac{a\lambda}{2}\right) + \frac{|a\lambda|}{2} = \left(1 - \frac{a\lambda}{2}\right) + \frac{a\lambda}{2} = 1 > 0
\]

In all cases the constant $(1 - \frac{a\lambda}{2}) - \frac{|a\lambda|}{2} > 0$, so that we divide by this positive constant to obtain stability.

Next we will consider the case $a\lambda > 1$. The technique is largely the same. After following the hint and rearranging, we find
\[
a\lambda^2 |v_{m+1}^{n+1}|^2 + \left(1 - \frac{a\lambda}{2}\right) v_{m+1}^{n+1} v_m + \left(1 - \frac{a\lambda}{2}\right) (v_{m+1}^{n+1} v_{m-1} - v_{m+1}^n v_{m}^n) = \frac{a\lambda}{2} |v_m^n|^2 + \left(1 - \frac{a\lambda}{2}\right) v_{m+1}^n v_{m-1}.
\]

After summing over $m$ and shifting the $m$ index on the last term of the left-hand side, we find
\[
\sum_{m \in \mathbb{Z}} \left[ \frac{a\lambda}{2} |v_{m+1}^{n+1}|^2 + \left(1 - \frac{a\lambda}{2}\right) v_{m+1}^{n+1} v_m \right] = \sum_{m \in \mathbb{Z}} \left[ \frac{a\lambda}{2} |v_m^n|^2 + \left(1 - \frac{a\lambda}{2}\right) v_{m+1}^n v_{m}^n \right]
\]
\[
\vdots
\]
\[
= \sum_{m \in \mathbb{Z}} \left[ \frac{a\lambda}{2} |v_0^0|^2 + \left(1 - \frac{a\lambda}{2}\right) v_0^0 v_0^0 \right].
\]

Just as before we have
\[
- \left|1 - \frac{a\lambda}{2}\right| (|v_m^n|^2 + |v_m^m|^2) \leq (1 - a\lambda) v_{m+1}^n v_{m}^m \leq \left|1 - \frac{a\lambda}{2}\right| (|v_{m+1}^{n+1}|^2 + |v_{m}^{n}|^2),
\]

and after summing and reindexing in $m$ as before, we obtain
\[
\left[ \frac{a\lambda}{2} - \left|1 - \frac{a\lambda}{2}\right| \right] \sum_{m \in \mathbb{Z}} |v_m^n|^2 \leq \left[ \frac{a\lambda}{2} + \left|1 - \frac{a\lambda}{2}\right| \right] \sum_{m \in \mathbb{Z}} |v_m^0|^2.
\]

This reduces exactly to the inequality obtained in the first part of the problem, if one defines a new parameter $\lambda'$ via
\[
\frac{a\lambda'}{2} := 1 - \frac{a\lambda}{2}.
\]

Furthermore, $a\lambda > 1$ implies that
\[
\frac{a\lambda'}{2} < \frac{1}{2} \implies a\lambda' < 1.
\]

Thus, the arguments of the first part imply that
\[
\frac{a\lambda}{2} - \left|1 - \frac{a\lambda}{2}\right| = \left(1 - \frac{a\lambda'}{2}\right) - \frac{|a\lambda'|}{2} > 0
\]
so that we divide by this positive constant to obtain stability.
Problem 5. Strikwerda 1.6.1.

The solution is in fact a direct consequence of Strikwerda 1.51, which was Problem 2 in this homework. Recall from Duhamel’s principle that the stability of the scheme for the inhomogeneous equation is equivalent to the stability of the scheme for the homogeneous equation, which may be written as

\[ v_{m+1}^n = \alpha v_{m+1}^n + \beta v_{m-1}^n \]

with

\[ \alpha = \left( \frac{1}{2} - \frac{a\lambda}{1 + (a\lambda)^2} \right), \quad \beta = \left( \frac{1}{2} + \frac{a\lambda}{1 + (a\lambda)^2} \right), \]

Because of the elementary inequalities \( 2x \leq 1 + x^2 \Rightarrow \frac{x}{1 + x^2} \leq \frac{1}{2} \), we see that \( \alpha \geq 0, \beta \geq 0, \) and \( |\alpha| + |\beta| = \alpha + \beta = 1. \) We have thus have unconditional stability, or stability for all values of \( a\lambda. \)

This is not a counterexample to Theorem 1.6.2, because the scheme is not generally consistent! In the same manner as Problem 1 we can find:

\[ P_{k,h}u = \partial_t u + \frac{2a}{1 + |a\lambda|^2} \partial_x u + \frac{1}{2} k\partial_t^2 u - \frac{h^2}{2k} \partial_x^2 u + \frac{ah^2}{3 + 3|a\lambda|^2} \partial_x^3 u + O(h^4 + k^{-1}h^4 + k^2). \]

This is only consistent with \( Pu = \partial_t u + a\partial_x u \) if \( |a\lambda| = 1, \) in which case the modified Lax-Friedrichs scheme reduces to the standard Lax-Friedrichs scheme.
Problem 6. Strikwerda 1.6.2.

It suffices to determine the “radius of dependence” on initial conditions and the rest of the proof is identical. For this, observe

\[ v_{m}^{n+1} = v_{m}^{n-1} + a\lambda (v_{m+1}^{n} - v_{m-1}^{n}), \quad n \geq 1 \]

To initialize this leapfrog scheme, one must use an explicit scheme for \( n = 1 \), e.g.

\[ v_{m}^{1} = v_{m}^{0} + \frac{a\lambda}{2} (v_{m+1}^{0} - v_{m-1}^{0}) \]

which we shall see in the next chapter preserves the 2nd-order accuracy of leapfrog. From this formula it is clear that

\[ v_{m}^{1} \in \text{Span}\{v_{m-1}^{0}, v_{m}^{0}, v_{m+1}^{0}\} \]

Continuing with the leapfrog iteration, we see that for \( n = 2 \)

\[ v_{m}^{2} = v_{m}^{0} + a\lambda (v_{m+1}^{1} - v_{m-1}^{1}) \in \text{Span}\{v_{m-2}^{0}, v_{m-1}^{0}, v_{m}^{0}, v_{m+1}^{0}, v_{m+2}^{0}\} \]

and, iteratively, increasing time-step \( n \) to \( n + 1 \), the “radius of dependence” in \( m \) increases by one step both directions. Thus

\[ v_{m}^{n+1} = \text{Span}\{v_{\ell}^{0} : \ell = m - n, \ldots, m, \ldots, m + n\} \]

As stated, the rest of the proof is identical, with \( x = \pm a \) being “out of reach” for this set of dependencies when \( |a\lambda| > 1 \) and thus preventing pointwise convergence.
Problem 7*. Prove the unconditional stability of the backward-time, central-space scheme

$$\frac{v_{m+1}^n - v_m^n}{k} + a \frac{v_{m+1}^{n+1} - v_{m-1}^{n+1}}{2h} = 0$$

by deriving and exploiting the following identity

$$\left(1 + \left(\frac{a\lambda}{2}\right)^2\right) \sum_m |v_{m+1}^{n+1}|^2 - \frac{(a\lambda)^2}{2} \sum_m v_{m+1}^{n+1} v_{m-1}^{n+1} = \sum_m |v_m^n|^2.$$

We can rewrite the scheme,

$$v_{m+1}^{n+1} + \frac{a\lambda}{2} (v_{m+1}^{n+1} - v_{m-1}^{n+1}) = v_m^n.$$

Thus, by squaring both sides,

$$|v_{m}^{n+1}|^2 + a\lambda v_m^{n+1} (v_{m+1}^{n+1} - v_{m-1}^{n+1}) + \frac{a^2\lambda^2}{4} (|v_{m+1}^{n+1}|^2 + |v_{m-1}^{n+1}|^2) - \frac{a^2\lambda^2}{2} v_{m+1}^{n+1} v_{m-1}^{n+1} = |v_m^n|^2.$$

Summing on both sides, we find

$$\left(1 + \frac{a^2\lambda^2}{2}\right) \sum_{m \in \mathbb{Z}} |v_{m+1}^{n+1}|^2 - \frac{a^2\lambda^2}{2} \sum_{m \in \mathbb{Z}} v_{m+1}^{n+1} v_{m-1}^{n+1} = \sum_{m \in \mathbb{Z}} |v_m^n|^2.$$

Recalling that $-p^2 - q^2 \leq -2pq$ for any $p, q$, we have

$$\sum_{m \in \mathbb{Z}} |v_{m}^{n+1}|^2 = \left(1 + \frac{a^2\lambda^2}{2}\right) \sum_{m \in \mathbb{Z}} |v_m^{n+1}|^2 - \frac{a^2\lambda^2}{4} \sum_{m \in \mathbb{Z}} (|v_{m+1}^{n+1}|^2 + |v_{m-1}^{n+1}|^2)$$

$$\leq \left(1 + \frac{a^2\lambda^2}{2}\right) \sum_{m \in \mathbb{Z}} |v_m^{n+1}|^2 - \frac{a^2\lambda^2}{2} \sum_{m \in \mathbb{Z}} v_{m+1}^{n+1} v_{m-1}^{n+1}$$

$$= \sum_{m \in \mathbb{Z}} |v_m^n|^2.$$

This can be stepped back in $n$ to give

$$\sum_{m \in \mathbb{Z}} |v_{m}^{n}|^2 \leq \sum_{m \in \mathbb{Z}} |v_0^{m}|^2.$$

for all $a\lambda$, which implies unconditional stability.