Problem 1. Consider the linear 4th-order PDE for the scalar function $u(x,y)$ on $\mathbb{R}^2$

$$au_{xxxx} + 2bu_{xxyy} + cu_{yyyy} = f(x,y).$$

(a) For all possible values of the real coefficients $a, b, c$, classify the type of this PDE as hyperbolic, parabolic, or elliptic.

(b) As a special case of your classification in (a), show that this equation is elliptic if $a > 0$, $c > 0$, and $b > -\sqrt{ac}$.

Problem 2. Show that the IVP for the quasilinear hyperbolic PDE

$$u_t + au_x = f(t,x,u), \quad u(0,x) = u_0(x) \quad (*)$$

is equivalent to the family of IVP's for the ODE's indexed by $\xi \in \mathbb{R}$

$$\frac{d\tilde{u}}{d\tau} = f(\tau, \xi + a\tau, \tilde{u}), \quad \tilde{u}(0, \xi) = u_0(\xi).$$

Show that the solution of (*) is given by $u(t,x) = \tilde{u}(t, x - at)$.

Problem 3. Use the method of Problem 2 to show that the solution of the IVP

$$u_t + u_x = \cos^2 u, \quad u(0,x) = u_0(x)$$

is given by $u(t,x) = \tan^{-1}\{\tan[u_0(x - t)] + t\}$.

Problem 4. Show that all solutions to

$$u_t + au_x = 1 + u^2$$

become unbounded in finite time. That is, $u(t,x)$ tends to infinity for $x = x_*$ as $t \to t_*$, where $t_*$ and $x_*$ are some finite values.

Problem 5*. Show that the initial value problem for the equation

$$u_t + (1 + x^2)u_x = 0$$

is not well-posed on the real line. Hint: Consider the region covered by the characteristics originating on the $x$-axis.
Problem 6. Solve the general initial value problem for
\[ u_t + \frac{1}{1 + \frac{1}{2} \cos x} u_x = 0. \]
Show that the solution is given by \( u(t, x) = u_0(\xi) \) where \( \xi \) is the unique solution of the nonlinear equation \( F(\xi, t, x) = \xi + \frac{1}{2} \sin \xi + t - x - \frac{1}{2} \sin x = 0 \). Use the implicit function theorem to show that this equation has a unique differentiable solution in the neighborhood of any point \((t, x)\).

Problem 7*. (a) Consider the IVP for the quasilinear hyperbolic PDE
\[ u_t + a(t, x)u_x = f(t, x, u), \quad u(0, x) = u_0(x) \]
and the associated family of IVP’s for the system of ODE’s indexed by \( \xi \in \mathbb{R} \)
\[ \frac{d\tilde{u}}{d\tau} = f(\tau, x(\tau), \tilde{u}), \quad \tilde{u}(0, \xi) = u_0(\xi) \]
\[ \frac{dx}{d\tau} = a(\tau, x(\tau)), \quad x(0) = \xi \]
(1)
(a) Show that the inverse \( \xi(\tau, x) \) at each fixed \( \tau \) of the function \( x(\tau, \xi) \) solves
\[ \xi_\tau + a(\tau, x)\xi_x = 0, \quad \xi(0, x) = x. \]
(b) Show that the solution of the initial-value problem for the hyperbolic PDE is given by \( u(t, x) = \tilde{u}(t, \xi(t, x)) \).
(c) Use the method of (b) to find the exact solution of the specific PDE
\[ u_t + \gamma xu_x = u \ln u \]
for general initial data \( u_0(x) \).