Homework No.6, 553.481/681, Due April 26, 2024.

Problem 1. [DOUBLE] (a) Try Euler's method with step length h on the simple test problem

$$\dot{y} = \lambda y, \quad y(0) = 1$$

for an arbitrary real parameter λ.

(i) Determine an explicit expression for y_n.

(ii) Determine for which values of λh the sequence $\{y_n\}_{n=0}^{\infty}$ is bounded.

(iii) Use $h = t/n$ to approximate the exact solution $y(t) = e^{\lambda t}$ by y_n. Show directly that $\lim_{n\to\infty} y_n = y(t)$ and, in fact, that

$$y_n - y(t) = -\frac{(\lambda t)^2}{2n} e^{\lambda t} + O\left(\frac{1}{n^2}\right). \qquad (*)$$

Hint: To show convergence, write $\left(1 + \frac{x}{n}\right)^n = \exp\left[n \ln\left(1 + \frac{x}{n}\right)\right]$ and use the Taylor expansion with remainder for $\ln(1 + x/n)$. To show the stronger result (*), use also the fact that

$$1 + \frac{x}{n} = e^{x/n} - \frac{1}{2}\left(\frac{x}{n}\right)^2 + O\left(\frac{1}{n^3}\right) = e^{x/n}\left[1 - \frac{1}{2}\left(\frac{x}{n}\right)^2 + O\left(\frac{1}{n^3}\right)\right],$$

because of the Taylor expansion with remainder for $e^{x/n}$.

(iv) From the result in (i) you can determine the local truncation error per step in the Euler method to be $\tau_n = [y(t_{n+1}; y_n) - y_{n+1}]/h = \frac{1}{2}\ddot{y}(t_n)h + O(h^2)$ for this situation, where $y(t; y_n)$ is the *local solution* that exactly solves the problem $\dot{y} = \lambda y$, $y(t_n) = y_n$. Show that the result obtained in (iii) is the same as $y_n - y(t) = -\delta(t)h + O(h^2)$, with prefactor $\delta(t)$ the solution of

$$\dot{\delta}(t) = \lambda\delta(t) + \frac{1}{2}\ddot{y}(t), \quad \delta(0) = 0. \qquad (**)$$

(b) Repeat (a) for Heun's method. In part (iii) you will need to find the replacement for (*) by generalizing the argument in (a). In part (iv) you should be able to show that $\tau_n = [y(t_{n+1}; y_n) - y_{n+1}]/h = \frac{1}{3!}\dddot{y}(t_n)h^2 + O(h^3)$ and now $y_n - y(t) = -\delta(t)h^2 + O(h^3)$, with $\delta(t)$ the solution of $\dot{\delta}(t) = \lambda\delta(t) + \frac{1}{3!}\dddot{y}(t)$.

Problem 2. (a) Consider the three stage Runge-Kutta formula

$$y_{n+1} = y_n + h[\gamma_1 K_1 + \gamma_2 K_2 + \gamma_3 K_3]$$

$$K_1 = f(t_n, y_n), \quad K_2 = f(t_n + \alpha_2 h, y_n + h\beta_{21} K_1)$$
$$K_3 = f(t_n + \alpha_3 h, y_n + h\beta_{31} K_1 + h\beta_{32} K_2)$$

Determine the set of equations that the coefficients $\{\gamma_j, \alpha_j, \beta_{ji}\}$ must satisfy if the formula is to be of order 3.

(b) The Runge-Kutta integration scheme with coefficients

$$\alpha_2 = 1/2, \quad \beta_{21} = 1/2$$

$$\alpha_3 = 3/4, \quad \beta_{31} = 0, \quad \beta_{32} = 3/4$$

$$\gamma_1 = 2/9, \quad \gamma_2 = 1/3, \quad \gamma_3 = 4/9$$

is employed as part of the Bogacki-Shampine algorithm implemented in MATLAB's integrator `ode23`. Use the result of part (a) to check whether this scheme satisfies the conditions to be third-order.

(c) Write a MATLAB script `rk3.m` to implement the scheme in part (b). Apply your code to solve the initial-value problem

$$\dot{y} = y \ln y, \quad y(0) = 2$$

with exact solution $Y(t) = 2^{\exp t}$ on the time-interval $[0, 1]$, using a number of steps $N = 4, 8, 16, ..., 2048$. Use the results to estimate the order of convergence. Does your numerical finding agree with your theoretical conclusion in part (b)?

Problem 3. (a) Use the Heun method to solve the the initial-value problem

$$y' = t^{1/5} y, \quad y(0) = 1$$

whose exact solution is $Y(t) = \exp(\frac{5}{6} t^{6/5})$. Solve the equation on $[0, 1]$, using a number of steps $N = 50, 100, 200, ..., 6400$ increasing by factors of 2. Calculate the base-2 logarithm of the ratios by which errors in $y(1)$ decrease for successive approximations when h is halved. How does this compare with the usual quadratic rate of convergence, which would yield 2 for this logarithm? Explain your results.

(b) Repeat part (a) using the classical 4th-order Runge-Kutta method.

Problem 4*. A function $f(t)$ is said to be $C^{k,\alpha}$ for integer $k \geq 0$ and $0 < \alpha < 1$ if it is C^k or k-times continuously differentiable and if also its kth-derivative is Hölder continuous with exponent α, that is:

$$|f^{(k)}(t) - f^{(k)}(t')| \leq K|t - t'|^{\alpha} \qquad (*)$$

for all t, t' with some constant K. In this problem we will explore the convergence of Runge-Kutta methods when the ODE function $f(t, y)$ is smooth in y but only $C^{k,\alpha}$ in t and when the solution $Y(t)$ is also only $C^{k+1,\alpha}$ in t.

(a) If a Runge-Kutta method is order n for ODE's with smooth solutions $Y(t)$, then under the above assumptions, prove that the method remains of order n for $k \geq n$ but might have order only $k + \alpha$ if $k < n$.

Hint: If $f(t)$ is $C^{k,\alpha}$, use the Hölder continuity condition (*) to derive a new error estimate on the Taylor polynomial of degree k.

(b) If, however, the assumption about $f(t, y)$ and $Y(t)$ holds only at $t = t_0$ and if these functions are C^{∞} for all $t > 0$, then prove that the Runge-Kutta method has order at least $k + \alpha + 1$ if $k < n$. Does this explain the results in Problem 3?

Hint: Consider the error made in the first step and then in all subsequent steps.

(c) As an example of the type discussed in part (a), consider the initial-value problem

$$\dot{y} = f(t), \ 0 < t < 1; \quad y(0) = 1/(1 - 2^{1+\alpha}),$$

with

$$f(t) = \sum_{n=1}^{\infty} \frac{\sin(2^n t)}{2^{\alpha n}}, \quad Y(t) = -\sum_{n=1}^{\infty} \frac{\cos(2^n t)}{2^{(1+\alpha)n}}.$$

The function f is a special case of the famous Weierstrass function which is $C^{0,\alpha}$ but not $C^{0,\beta}$ for any $\beta > \alpha$. As in Problem 3, use the Heun method to solve the initial-value problem for the case $\alpha = 1/2$ with number of steps $N = 200, ..., 51200$ increasing by factors of 2. Calculate the error $E(h) = \max_{t_n \in [0,1]} |y_n - Y(t_n)|$ for each value of h and make a straight line fit of $\log E(h)$ versus $\log h$ with `polyfit` in Matlab to determine the approximate order of convergence.

Is the observed order of convergence surprising, based on the analysis in (a)? Can you think of any explanation for your observation? Is there anything special about solving ODE's with $f(t, y)$ independent of y?