Problem 1. (a) Since $\alpha_i := \int_a^b \prod_{j \neq i} (x - x_j)/(x_i - x_j) \, dx$, we have

\[ \alpha_0 = \frac{1}{4!} \int_a^b (x - x_1)(x - x_2)(x - x_3)(x - x_4) \, dx \]

\[ \alpha_1 = -\frac{1}{3!} \int_a^b (x - x_0)(x - x_2)(x - x_3)(x - x_4) \, dx \]

\[ \alpha_2 = \frac{1}{2!2!} \int_a^b (x - x_0)(x - x_1)(x - x_3)(x - x_4) \, dx \]

\[ \alpha_3 = -\frac{1}{3!} \int_a^b (x - x_0)(x - x_1)(x - x_2)(x - x_4) \, dx \]

\[ \alpha_4 = \frac{1}{4!} \int_a^b (x - x_0)(x - x_1)(x - x_2)(x - x_3) \, dx. \]

From Program 1.1 we find that

\[ \alpha_0 = \frac{14h}{45}, \quad \alpha_1 = \frac{64h}{45}, \quad \alpha_2 = \frac{8h}{15}, \quad \alpha_3 = \frac{64h}{45}, \quad \alpha_4 = \frac{14h}{45}. \]

Recall that $f(x) = \sum_{i=0}^4 \alpha_i f(x_i) + f_0 x_0, \ldots, x_4, x \prod_{i=0}^4 (x - x_i)$. Apply integration by parts, the mean value theorem, and Program 1.1 again to write

\[ E_4(f) = \int_a^b f(x_0, \ldots, x_4, x) \frac{\mathrm{d}}{\mathrm{d}x} w(x) \, dx \]

\[ = -\int_a^b w(x) \frac{\mathrm{d}}{\mathrm{d}x} f(x_0, \ldots, x_4, x) \, dx \]

\[ = -\int_a^b f(x_0, \ldots, x_4, x, x) w(x) \, dx \]

\[ = -f(x_0, \ldots, x_4, x, x) \int_a^b w(x) \, dx \]

\[ = -\frac{f^{(6)}(x)}{6!} \int_a^b \int_0^x \prod_{i=0}^4 (x' - x_i) \, dx' \, dx \]

\[ = \frac{f^{(6)}(x)}{6!} \int_a^b (x - x_0) \prod_{i=0}^4 (x - x_i) \, dx \]

\[ = \frac{8h^7}{945} f^{(6)}(x). \]

It’s just algebra to see that assembling the collected $\alpha_i$, $f(x_i)$, and $E_4(f)$ fits the desired form.

(b) See Program 1.2.

(c) See Program 1.3 and its outputs Figure 1.1 and Figure 1.2 for the integral (i), and Program 1.4 and its outputs Figure 1.3 and Figure 1.4 for the integral (ii). Note the results for the composite Simpson’s rule are consistent with our expectations because its error is of order 4. By a similar argument to that of the composite Simpson’s rule error, we expect the error of the composite Boole’s rule to be of order 6. We see this is roughly the case, better observed for larger $n$.
Problem 2. (a) Write
\[
\sum_{j=1}^{\infty} B_j (1-x) \frac{(-t)^j}{j!} = -t \frac{e^{tx} - e^t}{e^t - 1}
\]
\[
= \frac{-t(e^{tx} - e^t)}{1 - e^t}
\]
\[
= \frac{t(e^{tx} - e^t)}{e^t - 1}
\]
\[
= \frac{t(e^{tx} - 1)}{e^t - 1} + \frac{t(1 - e^t)}{e^t - 1}
\]
\[
= \sum_{j=1}^{\infty} B_j(x) \frac{t^j}{j!} - t.
\]

Note the series expansion for \( t \) vanishes at the second and higher orders. Thus \( B_1(1-x) = 1 - B_1(x) \) and \( B_j(1-x) = B_j(x) \) for \( j \geq 2 \).

(b) Write
\[
\sum_{j=0}^{\infty} \frac{d}{dx} B_j(x) \frac{t^j}{j!} = \frac{d}{dx} \frac{t(e^{tx} - 1)}{e^t - 1}
\]
\[
= \frac{t^2 e^{tx}}{e^t - 1} = \frac{t^2(e^{tx} - 1)}{e^t - 1} + \frac{t^2}{e^t - 1}
\]
\[
= \sum_{j=1}^{\infty} [B_j(x) + B_j] \frac{t^{j+1}}{j!}
\]
\[
= \sum_{j=1}^{\infty} (j + 1) [B_j(x) + B_j] \frac{t^{j+1}}{(j+1)!}
\]
\[
= \sum_{j=2}^{\infty} j [B_{j-1}(x) + B_{j-1}] \frac{t^j}{j!}.
\]

For \( j \geq 2 \) this implies \( B'_{j}(x) = j[B_{j-1}(x) + B_{j-1}] \), as desired.

Problem 3. (a) Note that \( 3I(f) - 2I_M(f) - I_T(f) = O(h^4) \). Thus \( I(f) = [I_T(f) + 2I_M(f)]/3 + O(h^4) \)
where \( I_T(f) + 2I_M(f) = h[f(a) + f(b)]/2 + 2hf(c) = h[f(a) + 4f(c) + f(b)]/2 \). This is Simpson’s rule with increments of size \( h/2 \).

(b) Write composite trapezoidal with stepsize \( h_k \) as
\[
T^{(0)}(h_k) = \sum_{j=0}^{2^k-1} h_k \frac{f(x_j) + f(x_{j+1})}{2} = \sum_{j=0}^{2^k-1} h_k \frac{f(x_{2j}) + f(x_{2j+1}) + f(x_{2j+2})}{2},
\]
and with stepsize \( h_{k-1} = 2h_k \) as
\[
T^{(0)}(h_{k-1}) = \sum_{j=0}^{2^{k-1}-1} h_k [f(x_{2j}) + f(x_{2j+2})],
\]
so that combining them yields
which is composite Simpson. We expand

\[
\frac{1}{3} \sum_{j=0}^{2^{k-1}-1} h_k [f(x_{2j}) + 4f(x_{2j+1}) + f(x_{2j+2})] = \frac{1}{3} \sum_{j=0}^{2^{k-2}-1} h_k [f(x_{4j}) + 4f(x_{4j+1}) + 2f(x_{4j+2}) + 4f(x_{4j+3}) + f(x_{4j+4})],
\]

and write

\[
T^{(1)}(h_{k-1}) = \frac{1}{3} \sum_{j=0}^{2^{k-2}-1} 2h_k [f(x_{4j}) + 4f(x_{4j+2}) + f(x_{4j+4})],
\]

so that combining yields

\[
T^{(2)}(h_k) = \frac{16T^{(1)}(h_k) - T^{(1)}(h_{k-1})}{15} = \frac{2}{45} \sum_{j=0}^{2^{k-2}-1} h_k [7f(x_{4j}) + 32f(x_{4j+1}) + 12f(x_{4j+2}) + 32f(x_{4j+3}) + 7f(x_{4j+4})],
\]

which is composite Boole.

**Problem 4.** See Figure 4.1 for a plot of the integrand in (i) and Figure 4.2 for a plot of the integrand in (ii).

Running Program 4.1 for (i) evaluates the integral to about 0.7471 for both methods, with 1025 function calls in Romberg and 4841 in adaptive Simpson's. Clearly, Romberg is much superior to adaptive Simpson. This should be expected because the integrand is quite smooth, and Romberg works well to get highly accurate integrals of smooth functions.

Running Program 4.2 for (ii) evaluates the integral to about 0.7045 for both methods, with 16385 function calls in Romberg and 2409 in adaptive Simpson's. In contrast to the first case, adaptive Simpson is now greatly superior. The plot of the integrand shows very sharp jumps at \(x = \pm 1\), or “near-discontinuities” where derivatives are very large. Adaptive methods handle this type of integrand more efficiently, by using a small \(h\) only near the points \(x = \pm 1\), whereas Romberg with a uniform grid is forced to use a small \(h\) everywhere.

**Problem 5.** First, we write

\[
H_2(x) = f(a) + (x - a)f'(a) + (x - a)^2 f[a, a, b] + (x - a)^2(x - b)f[a, a, b, b],
\]

where

\[
f[a, a, b] = \frac{f[a, b] - f'(a)}{b - a}, \quad f[a, a, b, b] = \frac{f'(a) - 2f[a, b] + f'(b)}{(b - a)^2}.
\]

Thus

\[
H_2(x) := f(a) + sf'(a) + s^2 \left[ \frac{f(b) - f(a)}{h^2} - \frac{f'(a)}{h} \right] + s^2(s - h) \left[ \frac{f'(a) + f'(b)}{h^2} - \frac{2(f(b) - f(a))}{h} \right]
\]

\[
= \left[ 1 - \frac{s^2}{h^2} + \frac{2s^2(s - h)}{h^3} \right] f(a) + \left[ \frac{s^2}{h^2} - \frac{2s^2(s - h)}{h^3} \right] f(b) + \left[ s - \frac{s^2}{h} + \frac{s^2(s - h)}{h^2} \right] f'(a) + \frac{s^2(s - h)}{h^2} f'(b).
\]

Some algebra will get us our desired form:
\[
\begin{align*}
1 - \frac{s^2}{h^2} + \frac{2s^2(s-h)}{h^3} &= \frac{h^3 - 3hs^2 + 2s^3}{h^3}, \\
\frac{s^2}{h^2} - \frac{2s^2(s-h)}{h^3} &= \frac{3h^5 - 2s^2}{h^3}, \\
s - \frac{s^2}{h} + \frac{s^2(s-h)}{h^2} &= \frac{s[h^2 - 2hs + s^2]}{h^2}.
\end{align*}
\]

Let \(a = x_k\) and \(b = x_{k+1}\), then we're done. For computing the integral, use the obvious substitution \(x \mapsto s\) and write

\[
\int_{x_k}^{x_{k+1}} H_2(x) \, dx = \int_0^h H_2(s + x_0) \, ds
\]

\[
= \frac{h^4 - h^4/2}{h^3} y_{k+1} + \frac{h^4 - h^4 + h^4/2}{h^3} y_k + \frac{h^4/4 - h^4/3}{h^2} y_{k+1} + \frac{h^4/4 - 2h^4/3 + h^4/2}{h^2} y'_k
\]

\[
= \frac{h}{2} (y_{k+1} + y_k) - \frac{h^2}{12} (y_{k+1}' - y_k').
\]

**Problem 6.** (a) From the chapter on Newton’s divided differences we recall that

\[
P_\ell(x; x_0, \ldots, x_{\ell}) = \frac{(x - x_\ell)P_{\ell-1}(x; x_0, \ldots, x_{\ell-1}) + (x - x_0)P_{\ell-1}(x; x_1, \ldots, x_\ell)}{x_\ell - x_0}.
\]

Take \(x = 0\) and we’re done.

(b) Using part (a),

\[
P_\ell(0; h_{k-\ell}, \ldots, h_k) = \frac{hP_{\ell-1}(0; h_{k-\ell+1}, \ldots, h_k) - 2^{-\ell} h P_{\ell-1}(0; h_{k-\ell}, \ldots, h_{k-1})}{h - 2^{-\ell} h}
\]

\[
= \frac{2^\ell P_{\ell-1}(0; h_{k-\ell+1}, \ldots, h_k) - P_{\ell-1}(0; h_{k-\ell}, \ldots, h_{k-1})}{2^\ell - 1},
\]

where \(P_0(0; h_k) = T^{(0)}(h_k)\). We thus see that \(P_\ell(0; h_{k-\ell}, \ldots, h_k)\) and \(T^{(\ell)}(h_k)\) satisfy identical recursion relations in \(\ell\) and, furthermore, have identical starting values for \(\ell = 1\). Thus we conclude that

\[
P_\ell(0; h_{k-\ell}, \ldots, h_k) = T^{(\ell)}(h_k)
\]

This result gives a simple interpretation of the Romberg approximation \(T^{(\ell)}(h_k)\) as the result of using the polynomial interpolant on the data \((h_0, T^{(0)}(h_0)), \ldots, (h_\ell, T^{(0)}(h_\ell))\) to extrapolate to \(h = 0\).
Figure 1.1. Error in composite Simpson and linear fit with slope $-3.999$ for integral in (i).

Figure 1.2. Error in composite Boole and linear fit with slope $-6.1176$ for integral in (i).
Figure 1.3. Error in composite Simpson and linear fit with slope $-3.999$ for integral in (ii).

Figure 1.4. Error in composite Boole and linear fit with slope $-5.477$ for integral in (ii).
Figure 4.1. Plot of integrand $\sin(x^2)$ in (i).

Figure 4.2. Plot of integrand $\exp(-\exp(x^2))$ in (ii).
Program 1.1. Symbolic integration for $\alpha_i$ and $\int_a^b (x - x_0)^2 \prod_{i=1}^{l} (x - x_i) \, dx$ in Problem 1(a), with the transformation $x \mapsto (x - a)/h$. Note this substitution throws a factor of $h^5$ and $h^7$ in front respectively, which we omit.

```matlab
syms x
alp(1)=int((x-1)*(x-2)*(x-3)*(x-4),0,4)/factorial(4);
alp(2)=-int(x*(x-2)*(x-3)*(x-4),0,4)/factorial(3);
alp(3)=int(x*(x-1)*(x-3)*(x-4),0,4)/factorial(2)^2;
alp(4)=-int(x*(x-1)*(x-2)*(x-4),0,4)/factorial(3);
alp(5)=int(x*(x-1)*(x-2)*(x-3),0,4)/factorial(4)
bet=int(x^2*(x-1)*(x-2)*(x-3)*(x-4),0,4)/factorial(6)
```

Program 1.2. Implementation of boolc.m.

```matlab
function I=boolc(f,a,b,n)
if mod(n,4)~=0
    'n not a multiple of 4 in Boole'
    return
end
h=(b-a)/n;
x1=a+h:4*h:b-3*h;
x2=a+2*h:4*h:b-2*h;
x3=a+3*h:4*h:b-h;
x4=a+4*h:4*h:b-4*h;
I=7*(f(a)+f(b))+32*(sum(f(x1))+sum(f(x3)))+12*sum(f(x2))+14*sum(f(x4));
I=2*h*I/45.;
end
```

Program 1.3. Comparisons of log-log-error for Simpson’s and Boole’s methods for integral (i).

```matlab
f=@(x) exp(-x.*2); a=0; b=1;
It=sqrt(pi)*erf(1)/2;
for ii=1:6
    n=4*2^ii;
nn(ii)=n;
simp(ii)=simpc(f,a,b,n);
bool(ii)=boolc(f,a,b,n);
end
eerrs=abs(simp-It); errb=abs(bool-It);
figure; Ps=polyfit(log(nn),log(errs),1);
ords=-Ps(1)
loglog(nn,errs,'-b',nn,exp(Ps(1)*log(nn)+Ps(2)),'--r')
```

```matlab
f=@(x) 1./(1+x .^2); a=-4; b=4;
It =2* atan (4);
for ii =1:8
    n=4*2^ ii;
    nn(ii )=n;
    simp(ii )= simpc (f,a,b,n);
    bool(ii )=boolc (f,a,b,n);
end
I= simp (8)
Ib= bool (8)
errs = abs (simp -It ); errb = abs (bool -It );
figure; Ps= polyfit (log (nn(5:8)) ,log (errs (5:8)) ,1);
ords =-Ps (1)
loglog (nn,errs ,'-b',nn,exp (Ps (1)* log (nn)+Ps (2)) ,'-r')
legend (','fit')
xlabel ('n'); ylabel ('err ')
title ([ (ii) Error in Composite Simpson and Linear Fit with Slope = -', num2str ( ords )])
```

Program 4.1. Romberg integration versus adaptive extrapolated Simpson’s rule for (i).

```matlab
f=@(x) sin (x .^2);
fplot(f ,[0 4])
xlabel('$x$ ',' FontSize ' ,15, ' Interpreter ','latex ');
ylabel ('$f(x)$',' FontSize ' ,15, ' Interpreter ','latex ')
title ('(i) $f(x)=\sin(x^2)$ ',' FontSize ' ,18, ' Interpreter ','latex ')
[IR,fcountR]=romberg (f,0,4,1e-14)
[IQ,fcountQ]=quad (f,0,4,1e-14)
```
Program 4.2. Romberg integration versus adaptive extrapolated Simpson’s rule for (ii).

```
g=@(x) exp(-exp(x.^20));
figure; fplot(g,[-2 2])
xlabel('x',' FontSize ' ,15, ' Interpreter ','latex ');
ylabel('f(x)',' FontSize ' ,15, ' Interpreter ','latex ')
title('(ii) $f(x) = \exp(-\exp(x^{20}))$',' FontSize ' ,18, ' Interpreter ','latex ')
[IR,fcountR]=romberg(g,-2,2,1e-14)
[IQ,fcountQ]=quad(g,-2,2,1e-14)
```