Problem 1. [DOUBLE] (a) Derive Boole’s rule, the Newton-Cotes quadrature rule for \( n = 4 \), so that, with \( x_i = a + ih, \ i = 0, \ldots, 4 \) for \( h = (b - a)/4 \), and \( \xi \in [a, b] \)

\[
\int_a^b f(x) \, dx = \frac{2h}{45} \left[ 7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4) \right] - \frac{8h^7}{945} f^{(6)}(\xi)
\]

You do not need to evaluate by hand the integrals to determine the rational coefficients \( \alpha_i, \ i = 0, 1, \ldots, 4 \) but instead you can use symbolic integration (for example, \texttt{int} in Matlab). To evaluate the error term, you should derive and use the expression

\[
E_4(f) = \int_a^b f[x_0, x_1, x_2, x_3, x_4, x] \prod_{i=0}^4 (x - x_i)
\]

Then you can use without proof the fact that \( w(x) := \int_a^x \prod_{i=0}^4 (\bar{x} - x_i) \, d\bar{x} \geq 0 \). The integral to determine the error coefficient can be again obtained symbolically.

(b) Modify the course script \texttt{simp.c.m} to write a code \texttt{boolc.m} that implements the composite Boole’s rule.

(c) Use your code to compare the composite Simpson’s rule and composite Boole’s rule applied to the following two integrals:

\[
(i) \quad \int_0^1 dx \sin(-\pi x^2/2) \quad (ii) \quad \int_{-4}^4 dx \frac{1}{(1+x^2)^{3/2}}.
\]

Use \( n = 8, 16, 32, 64, 128, 256 \) and make a log-log plot of the errors in the approximations versus \( n \). Are the results consistent with the proven asymptotic order of convergence? Explain your answer. For (ii) it may help to consider even larger \( n \).

\textit{Note:} The exact value of integral (i) can be obtained from the Matlab function \texttt{fresnels} and integral (ii) is \( \int_{-c}^c dx \frac{1}{(1+x^2)^{3/2}} = 2c/\sqrt{1+c^2} \).

Problem 2. (a) Prove that the Bernoulli polynomials satisfy the following reflection property:

\[
(-1)^j B_j(1 - x) = B_j(x), \quad j \geq 2.
\]

(b) Prove the following identity relating the Bernoulli polynomials and Bernoulli numbers:

\[
B'_j(x) = j[B_{j-1}(x) + B_{j-1}], \quad j \geq 2.
\]

\textit{Note:} These results can be used to give a general proof of the Euler-MacLaurin formula. See Ralston, \textit{A First Course in Numerical Analysis} (McGraw-Hill, 1965).
**Problem 3.** (a) The composite midpoint rule \( I_n^M(f) \) with \( h = (b-a)/n \) for evaluating the integral \( I(f) = \int_a^b f(x) \, dx \) can be shown to have the asymptotic error formula

\[
I(f) = I_n^M(f) - \frac{h^2}{24} [f'(b) - f'(a)] + O(h^4).
\]

Using this information, obtain a new numerical integration formula \( \tilde{I}_n(f) \) with a higher order of convergence by making a linear combination of \( I_n^M(f) \) and the composite trapezoidal rule \( I_n^T(f) \). Write out the weights for this new formula \( \tilde{I}_n(f) \),

(b) Show that in Romberg integration, with \( T^{(0)}(h_k) \) the composite trapezoidal rule,

\[
T^{(1)}(h_k) = \frac{1}{3} [4T^{(0)}(h_k) - T^{(0)}(h_{k-1})]
\]

is the composite Simpson rule and

\[
T^{(1)}(h_k) = \frac{1}{15} [16T^{(1)}(h_k) - T^{(1)}(h_{k-1})]
\]

is the composite Boole rule.

**Problem 4.** Use the MATLAB script `romberg.m` and the intrinsic function `quad` in order to compare Romberg integration and adaptive extrapolated Simpson’s rule applied to the following integrals:

(i) \( \int_0^2 dx \, \exp(-x^4) \) \hspace{1cm} (ii) \( \int_0^2 dx \, \exp(-x^4/(1-x^4)) \text{sign}(1-|x|) \)

Calculate each integral to a tolerance of \( tol = 10^{-14} \) and record the number of function calls made by both algorithms. Explain your results by plotting the integrands and invoking their geometric and smoothness properties.

**Problem 5*. Show that the cubic Hermite interpolating polynomial on the interval \( x_k < x < x_{k+1} \) is given by

\[
H_2(x) = \frac{3h^2s^2 - 2s^3}{h^3} y_{k+1} + \frac{h^3 - 3hs^2 + 2s^3}{h^3} y_k + \frac{s^2(s-h)}{h^2} y'_{k+1} + \frac{s(s-h)^2}{h^2} y'_k
\]

with \( s = x - x_k \) and \( y_i = f(x_i) \), \( y'_i = f'(x_i) \), \( i = k, k+1 \) and that its integral over that interval is given by

\[
\int_{x_k}^{x_{k+1}} dx \, H_2(x) = \frac{h}{2} (y_{k+1} + y_k) - \frac{h^2}{12} (y'_{k+1} - y'_k).
\]

Thus, integrating the piecewise cubic Hermite interpolating polynomial yields the composite corrected trapezoidal rule.
**Problem 6**. Show that if \( h_k = 2^{-k}h \), then the Romberg approximate integrators are given by

\[
T^\ell(h_k) = p_\ell(0; h_{k-\ell}, ..., h_k), \quad k \geq \ell
\]

where the polynomials interpolate the data \((h_k, T^{(0)}(h_k)), k = 0, 1, 2, ...\)

*Hint:* Use the result proved in class: if a set of data \((x_i, y_i), i = 0, 1, 2, ...\) are given, then one can inductively define \( p_\ell(x; x_0, ..., x_\ell) \), the polynomial of degree \( \leq \ell \) that interpolates the data on the points \( x_0, ..., x_\ell \), by the formula

\[
 p_\ell(0; x_0, ..., x_\ell) = \frac{x_0 p_{\ell-1}(0; x_1, ..., x_\ell) - x_\ell p_{\ell-1}(0; x_0, ..., x_{\ell-1})}{x_0 - x_\ell}.
\]