Homework No.4, 553.481/681, Due March 29, 2024.

Problem 1. [DOUBLE] (a) Derive Boole's rule, the Newton-Cotes quadrature rule for $n = 4$, so that, with $x_i = a + ih$, $i = 0, .., 4$ for $h = (b - a)/4$, and $\xi \in [a, b]$

$$\int_a^b f(x)dx = \frac{2h}{45}\left[7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)\right] - \frac{8h^7}{945}f^{(6)}(\xi)$$

You do not need to evaluate by hand the integrals to determine the rational coefficients α_i, $i = 0, 1, ..., 4$ but instead you can use symbolic integration (for example, `int` in Matlab). To evaluate the error term, you should derive and use the expression

$$E_4(f) = \int_a^b f[x_0, x_1, x_2, x_3, x_4, x] \prod_{i=0}^{4}(x - x_i)$$

Then you can use without proof the fact that $w(x) := \int_a^x \prod_{i=0}^4 (\bar{x} - x_i)\, d\bar{x} \geq 0$. The integral to determine the error coefficient can be again obtained symbolically.

(b) Modify the course script `simpc.m` to write a code `boolc.m` that implements the composite Boole's rule.

(c) Use your code to compare the composite Simpson's rule and composite Boole's rule applied to the following two integrals:

(i) $\int_0^1 dx\ \sin(-\pi x^2/2)$ *(ii)* $\int_{-4}^4 dx\ \frac{1}{(1+x^2)^{3/2}}$.

Use $n = 8, 16, 32, 64, 128, 256$ and make a log-log plot of the errors in the approximations versus n. Are the results consistent with the proven asymptotic order of convergence? Explain your answer. For *(ii)* it may help to consider even larger n.

Note: The exact value of integral *(i)* can be obtained from the Matlab function `fresnels` and integral *(ii)* is $\int_{-c}^{c} dx\ \frac{1}{(1+x^2)^{3/2}} = 2c/\sqrt{1 + c^2}$.

Problem 2. (a) Prove that the Bernoulli polynomials satisfy the following reflection property:

$$(-1)^j B_j(1 - x) = B_j(x), \ \ j \geq 2.$$

(b) Prove the following identity relating the Bernoulli polynomials and Bernoulli numbers:

$$B_j'(x) = j[B_{j-1}(x) + B_{j-1}], \ \ j \geq 2.$$

Note: These results can be used to give a general proof of the Euler-MacLaurin formula. See Ralston, *A First Course in Numerical Analysis* (McGraw-Hill, 1965).

Problem 3. (a) The composite midpoint rule $I_n^M(f)$ with $h = (b-a)/n$ for evaluating the integral $I(f) = \int_a^b f(x)\,dx$ can be shown to have the asymptotic error formula

$$I(f) = I_n^M(f) - \frac{h^2}{24}[f'(b) - f'(a)] + O(h^4).$$

Using this information, obtain a new numerical integration formula $\tilde{I}_n(f)$ with a higher order of convergence by making a linear combination of $I_n^M(f)$ and the composite trapezoidal rule $I_n^T(f)$. Write out the weights for this new formula $\tilde{I}_n(f)$,

(b) Show that in Romberg integration, with $T^{(0)}(h_k)$ the composite trapezoidal rule,

$$T^{(1)}(h_k) = \frac{1}{3}[4T^{(0)}(h_k) - T^{(0)}(h_{k-1})$$

is the composite Simpson rule and

$$T^{(1)}(h_k) = \frac{1}{15}[16T^{(1)}(h_k) - T^{(1)}(h_{k-1})$$

is the composite Boole rule.

Problem 4. Use the `MATLAB` script `romberg.m` and the intrinsic function `quad` in order to compare Romberg integration and adaptive extrapolated Simpson's rule applied to the following integrals:

(i) $\int_0^2 dx\ \exp(-x^4)$ *(ii)* $\int_0^2 dx\ \exp(-x^4/(1-x^4))\text{sign}(1-|x|)$

Calculate each integral to a tolerance of $tol = 10^{-14}$ and record the number of function calls made by both algorithms. Explain your results by plotting the integrands and invoking their geometric and smoothness properties.

Problem 5*. Show that the cubic Hermite interpolating polynomial on the interval $x_k < x < x_{k+1}$ is given by

$$H_2(x) = \frac{3hs^2 - 2s^3}{h^3}y_{k+1} + \frac{h^3 - 3hs^2 + 2s^3}{h^3}y_k + \frac{s^2(s-h)}{h^2}y'_{k+1} + \frac{s(s-h)^2}{h^2}y'_k$$

with $s = x - x_k$ and $y_i = f(x_i),\ y'_i = f'(x_i),\ i = k, k+1$ and that its integral over that interval is given by

$$\int_{x_k}^{x_{k+1}} dx\, H_2(x) = \frac{h}{2}(y_{k+1} + y_k) - \frac{h^2}{12}(y'_{k+1} - y'_k).$$

Thus, integrating the piecewise cubic Hermite interpolating polynomial yields the composite corrected trapezoidal rule.

Problem 6*. Show that if $h_k = 2^{-k}h$, then the Romberg approximate integrators are given by

$$T^{(\ell)}(h_k) = p_\ell(0; h_{k-\ell}, ..., h_k), \quad k \geq \ell$$

where the polynomials interpolate the data $(h_k, T^{(0)}(h_k))$, $k = 0, 1, 2, ...$

Hint: Use the result proved in class: if a set of data (x_i, y_i), $i = 0, 1, 2, ...$ are given, then one can inductively define $p_\ell(x; x_0, ..., x_\ell)$, the polynomial of degree $\leq \ell$ that interpolates the data on the points $x_0, ..., x_\ell$, by the formula

$$p_\ell(0; x_0, ..., x_\ell) = \frac{x_0 p_{\ell-1}(0; x_1, ..., x_\ell) - x_\ell p_{\ell-1}(0; x_0, ..., x_{\ell-1})}{x_0 - x_\ell}.$$