Problem 1. [DOUBLE] (a) Derive Boole’s rule, the Newton-Cotes quadrature rule for $n = 4$, so that, with $x_i = a + ih, i = 0, \ldots, 4$ for $h = (b - a)/4$, and $\xi \in [a, b]$

$$\int_a^b f(x)dx = \frac{2h}{45} \left[ 7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4) \right] - \frac{8h^7}{945} f^{(6)}(\xi)$$

You do not need to evaluate by hand the integrals to determine the rational coefficients $\alpha_i, i = 0, 1, \ldots, 4$ but instead you can use symbolic integration (for example, int in Matlab). To evaluate the error term, you should derive and use the expression

$$E_4(f) = \int_a^b f[x_0, x_1, x_2, x_3, x_4, x] \prod_{i=0}^4 (x - x_i)$$

Then you can use without proof the fact that $w(x) := \int_a^x \prod_{i=0}^4 (\bar{x} - x_i) d\bar{x} \geq 0$. The integral to determine the error coefficient can be again obtained symbolically.

(b) Modify the course script simp.m to write a code boolc.m that implements the composite Boole’s rule.

(c) Use your code to compare the composite Simpson’s rule and composite Boole’s rule applied to the following two integrals:

(i) $\int_0^1 dx \exp(-x^2)$

(ii) $\int_{-4}^4 dx \frac{1}{1+x^2}$.

Use $n = 8, 16, 32, 64, 128, 256$ and make a log-log plot of the errors in the approximations versus $n$. Are the results consistent with the proven asymptotic order of convergence? Explain your answer. For (ii) it may help to consider even larger $n$.

Note: The exact value of integral (i) can be obtained from the Matlab function erf and integral (ii) is $\int_{-4}^4 dx \frac{1}{1+x^2} = 2 \arctan(4)$.

Problem 2. (a) Prove that the Bernoulli polynomials satisfy the following reflection property:

$$(-1)^j B_j(1 - x) = B_j(x), \quad j \geq 2.$$

(b) Prove the following identity relating the Bernoulli polynomials and Bernoulli numbers:

$$B'_j(x) = j[B_{j-1}(x) + B_{j-1}], \quad j \geq 2.$$

Note: These results can be used to give a general proof of the Euler-MacLaurin formula. See Ralston, *A First Course in Numerical Analysis* (McGraw-Hill, 1965).
Problem 3. (a) The midpoint rule \( I_M(f) = hf \left( \frac{a+b}{2} \right) \), \( h = b - a \) for evaluating the integral \( I(f) = \int_a^b f(x) \, dx \) can be shown to have the asymptotic error formula

\[
I(f) = I_M(f) + \frac{h^2}{24}[f'(b) - f'(a)] + O(h^4).
\]

Using this information, obtain a new numerical integration formula \( \tilde{I}(f) \) with a higher order of convergence by making a linear combination of \( I_M(f) \) and the trapezoidal rule \( I_T(f) = \frac{h}{2}[f(a) + f(b)] \). Write out the weights for this new formula \( \tilde{I}(f) \),

(b) Show that in Romberg integration, with \( T^{(0)}(h_k) \) the composite trapezoidal rule,

\[
T^{(1)}(h_k) = \frac{1}{3}[4T^{(0)}(h_k) - T^{(0)}(h_{k-1})]
\]

is the composite Simpson rule and

\[
T^{(1)}(h_k) = \frac{1}{15}[16T^{(1)}(h_k) - T^{(1)}(h_{k-1})]
\]

is the composite Boole rule.

Problem 4. Use the MATLAB script \texttt{romberg.m} and the intrinsic function \texttt{quad} in order to compare Romberg integration and adaptive extrapolated Simpson’s rule applied to the following integrals:

(i) \( \int_0^4 dx \sin(x^2) \)  
(ii) \( \int_{-2}^2 dx \exp[-\exp(x^{20})] \).

Calculate each integral to a tolerance of \( tol = 10^{-14} \) and record the number of function calls made by both algorithms. Explain your results, using the geometric and smoothness properties of the integrands.

Problem 5*. Show that the cubic Hermite interpolating polynomial on the interval \( x_k < x < x_{k+1} \) is given by

\[
H_2(x) = \frac{3hs^2 - 2s^3}{h^3}y_{k+1} + \frac{h^3 - 3hs^2 + 2s^3}{h^3}y_k + \frac{s^2(s - h)}{h^2}y'_{k+1} + \frac{s(s - h)^2}{h^2}y'_k
\]

with \( s = x - x_k \) and \( y_i = f(x_i), \ y_i' = f'(x_i), \ i = k, k + 1 \) and that its integral over that interval is given by

\[
\int_{x_k}^{x_{k+1}} dx H_2(x) = \frac{h}{2}(y_{k+1} + y_k) - \frac{h^2}{12}(y'_{k+1} - y'_k).
\]

Thus, integrating the piecewise cubic Hermite interpolating polynomial yields the composite corrected trapezoidal rule.
Problem 6*. (a) If a set of data \((x_i, y_i), i = 0, 1, 2, ...\) are given, prove inductively that
\[
p_\ell(0; x_0, ..., x_\ell) = \frac{x_0 p_{\ell-1}(0; x_1, ..., x_\ell) - x_\ell p_{\ell-1}(0; x_0, ..., x_{\ell-1})}{x_0 - x_\ell}
\]
where \(p_\ell(x; x_0, ..., x_\ell)\) is the polynomial of degree \(\leq \ell\) that interpolates the data on the points \(x_0, ..., x_\ell\).

(b) Use part (a) to show that if \(h_k = 2^{-k}h\), then the Romberg approximate integrators are given by
\[
T^{(\ell)}(h_k) = p_\ell(0; h_{k-\ell}, ..., h_k), \quad k \geq \ell
\]
where the polynomials interpolate the data \((h_k, T^{(0)}(h_k)), k = 0, 1, 2, ...\)