Problem 1. For each of the numbers $x$ given below, answer the following three questions:

(i) What is the representation of the number $x$ in base-2? Write the number in the form $x = (1 + F) \times 2^E$ where the fraction $0 \leq F < 1$ should be given its base-2 representation and the exponent $E$ is an integer.

(ii) Is $x$ a machine number in IEEE standard single precision arithmetic? Explain why or why not, using the results of (i). (Only arguments using (i) will be accepted and other explanations, even if correct, will receive no credit. You are welcome to use Matlab to check your answers, but no numerical results will be accepted as answers.)

(iii) If $x$ is not a machine number, what machine number $\hat{x}$ is used to represent it in IEEE standard single precision arithmetic if using the “round-to-even” rule? Give the result as $\hat{x} = (1 + \hat{F}) \times 2^{\hat{E}}$ with $\hat{F}$ in base-2 for normalized numbers, as $\hat{x} = \hat{F} \times 2^{\hat{E}}$ and $\hat{F}$ in base-2 for denormalized numbers, or else as INF, NAN, etc.

\[ \begin{array}{llllll}
(2a) & x = 8 & (2b) & x = 1/8 & (2c) & x = 5 \\
(2e) & x = 2^{24} & (2f) & x = 2^{24} + 1 & (2g) & x = 1 + 2^{-23} \\
(2i) & x = 2^{-149} & (2j) & x = 2^{-150} & (2k) & x = 2^{128} - 2^{104} \\
(2l) & x = 2^{128} - 2^{103} & \end{array} \]
Problem 2. This problem concerns the symmetric-difference approximation to the second-derivative

\[ D_h^2 f(x) = f(x + h) + f(x - h) - 2f(x) \]

\[ \frac{\text{h}^2}{h^2} \stackrel{\text{def}}{=} f''(x), \quad h > 0. \]

(a) Use the Taylor expansion with remainder to prove that for any 4 times continuously differentiable function \( f \)

\[ D_h^2 f(x) = f''(x) + \frac{1}{12} f^{(4)}(\xi) h^2, \quad \xi \in [x - h, x + h]. \]

\hspace{1cm} \text{(*)}

\text{Hint: The Intermediate Value Theorem is needed to prove that } (f^{(m)}(\xi_1) + f^{(m)}(\xi_2))/2 = f^{(m)}(\xi) \text{ for some } \xi \in [\xi_1, \xi_2].

(b) When \( f(x) = \frac{1}{6} x^3 \), use part (a) to show that there is no error in \( D_h^2 f(x) \) for any \( h > 0 \), in which case one has

\[ D_h^2 f(x) = x = f''(x) \quad \text{for all } h > 0, \]

in exact arithmetic. Evaluate the approximation \( D_h^2 f(x) \) numerically in Matlab with single-precision arithmetic for \( f(x) = \frac{1}{6} x^3 \), \( x = 2 \), and for the values \( h = 10^{1-i/2}, i = 1, 2, 3, \ldots, 11 \). Give the relative error for each choice of \( h \).

\text{Remark: Defining in Matlab } x = \text{single}(2) \text{ and } h = \text{single}(10.^(1-i/2)), \text{ all of the remaining calculations will automatically be carried out in single precision.}

(c) Are the results in (b) accurate to single precision? Why or why not? What is the source of the error?

(d) When \( f(x) = \frac{1}{8} x^4 \), use equation (*) in obtain an optimal value \( h_* \) for which the error \( Err_* \) is minimum in single precision arithmetic.

(e) For \( f(x) = \frac{1}{6} x^3 \) and for the same values \( x = 2 \) and \( h = 10^{1-i/2}, i = 1, 2, 3, \ldots, 11 \) as in (b), evaluate the approximation \( D_h^2 f(x) \) numerically in Matlab with single precision arithmetic. Give the relative error \( Rel \) for each \( h \) and make a log-log plot of \( Rel \) vs. \( h_* \), together with the theoretical values \( Rel_*, h_* \) from (d).
Problem 3. Consider each of the following three initial-value problems:

\[(i) \quad \dot{x} = x, \quad x(0) = x_0\]
\[(ii) \quad \dot{x} = x \ln x, \quad x(0) = x_0\]
\[(iii) \quad \dot{x} = x^2, \quad x(0) = x_0\]

and answer for each of them the following questions:

(a) Is the problem well-posed forward in time? If so, find the unique solution \(x(t; x_0)\) and verify that it is continuous in the data \(x_0\). If not, state what new conditions must be added to make the problem well-posed.

(b) Calculate the (relative) condition number of the solution \(x(t)\) for infinitesimal changes of the input \(x_0\). Under what restrictions is the problem well-conditioned?

(c) Now consider the inverse problem, that is, the problem to determine \(x_0\) given the value \(x\) of the solution \(x(t)\) at time \(t\). Are these problems well-posed for general real values of \(x\) and times \(t\)? If not, state the restrictions needed to make the problem well-posed and find the unique solutions \(x_0(x; t)\) which are continuous in \(x\).

(d) Calculate the condition numbers of \(x_0(x; t)\) for infinitesimal changes in \(x\). Under what restrictions is the inverse problem well-conditioned?
Problem 4. This problem studies the use of Steffensen’s method, which introduces a new iteration function

\[ \hat{g}(x) = g(g(x)) - \frac{(g(g(x)) - g(x))^2}{g(g(x)) - 2g(x) + x} \]

with improved convergence relative to fixed-point iteration with function \( g(x) \).

(a) [681 students only] If \( g(x) \) is a \( C^2 \) function with fixed point \( x_* = g(x_*) \) and asymptotic linear convergence rate \( \lambda_* = g'(x_*) \neq 1 \), then prove that iteration with \( \hat{g}(x) \) has at least quadratic rate of convergence. Note that Steffensen’s method thus converts even a linearly divergent iteration to a quadratically convergent iteration, unless \( \lambda_* = g'(x_*) = 1 \! \).

(b) Although we shall not prove it, a vectorized form of Steffensen’s method also improves convergence of fixed-point iterations with an \( n \)-dimensional vector function \( \mathbf{g}(x) \), where now

\[ \hat{g}_i = g_i(\mathbf{g}(x)) - \frac{(g_i(\mathbf{g}(x)) - g_i(x))^2}{g_i(\mathbf{g}(x)) - 2g_i(x) + x_i}, \quad i = 1, \ldots, n. \]

(c) What is the expected rate of convergence of the Steffensen-improved Newton iteration? Let \( \tau_f \) be the time required for one evaluation of both \( \mathbf{f}(x) \) and \( \mathbf{Df}(x) \). Calculate the expected number of iterations \( N_{\text{steff}} \) and the total time \( T_{\text{steff}} \) for Steffensen-improved Newton method to converge to given tolerance \( \epsilon \ll 1 \) starting at initial distance \( |x_0 - x_*| \), in terms of \( \tau_f \) and the constant \( K = \log(M\epsilon)/\log(M|x_0 - x_*|) \), where \( M \) is the constant used in proof of convergence of the Newton method. Compare this with the number of iterations \( N_{\text{newt}} \) and the total time \( T_{\text{newt}} \) expected for the original Newton method. Which is expected to be faster, in exact arithmetic and asymptotically for \( \epsilon \ll 1 \)?

(d) Apply both the Newton-Raphson code \texttt{newtraph.m} and your code \texttt{steffnewt.m} from part (b) to find the root of the vector function

\[ \mathbf{f}(x) = [x_1^3 + x_1^2 x_2 - x_1 x_3 + 6, \exp(x_1) + \exp(x_2) - x_3, x_2^2 - 2x_1 x_3 - 4]^T, \]

with initial guess \( x_0 = [-2, 0, 1]^T \) and requested tolerance \( \epsilon = 10^{-15} \). What is the number of iterations required for each method? Make also a careful comparison of the wall clock time for each method. Which is faster?

(e) Repeat part (d) for the scalar function

\[ f(x) = (x - e) \ln(x/e) \]

with initial guess \( x_0 = 3 \). What accounts for the dramatic difference from part (d) in the performance of Newton’s method?