Observe that Vogel's method avoids the costly shipments associated with $x_{22}$ and $x_{31}$. This is because the high shipping costs resulted in large penalties that caused Vogel's method to choose other variables to satisfy the second and third demand constraints.

Of the three methods we have discussed for finding a bfs, the northwest corner method requires the least effort and Vogel's method requires the most effort. Extensive research [Glover et al. (1974)] has shown, however, that when Vogel's method is used to find an initial bfs, it usually takes substantially fewer pivots than if the other two methods had been used. For this reason, the northwest corner and minimum-cost methods are rarely used to find a basic feasible solution to a large transportation problem.

**PROBLEMS**

**Group A**

1. Use the northwest corner method to find a bfs for Problems 1, 2, and 3 of Section 7.1.

2. Use the minimum cost method to find a bfs for Problems 4, 7, and 8 of Section 7.1. (Hint: For a maximization problem, call the minimum-cost method the maximum-profit method or the maximum-revenue method.)

3. Use Vogel's method to find a bfs for Problems 5 and 6 of Section 7.1.

4. How should Vogel's method be modified to solve a maximization problem?

---

### 7.3 The Transportation Simplex Method

In this section, we show how the simplex algorithm simplifies when a transportation problem is solved. We begin by discussing the pivoting procedure for a transportation problem.

Recall that when the pivot row was used to eliminate the entering basic variable from other constraints and row 0, many multiplications were usually required. In solving a transportation problem, however, pivots require only additions and subtractions.

**How to Pivot in a Transportation Problem**

By using the following procedure, the pivots for a transportation problem may be performed within the confines of the transportation tableau:

**Step 1** Determine (by a criterion to be developed shortly) the variable that should enter the basis.

**Step 2** Find the loop (it can be shown that there is only one loop) involving the entering variable and some of the basic variables.

**Step 3** Counting only cells in the loop, label those found in Step 2 that are an even num-
ber (0, 2, 4, and so on) of cells away from the entering variable as even cells. Also label those that are an odd number of cells away from the entering variable as odd cells.

**Step 4** Find the odd cell whose variable assumes the smallest value. Call this value \( \theta \). The variable corresponding to this odd cell will leave the basis. To perform the pivot, decrease the value of each odd cell by \( \theta \) and increase the value of each even cell by \( \theta \). The values of variables not in the loop remain unchanged. The pivot is now complete. If \( \theta = 0 \), then the entering variable will equal 0, and an odd variable that has a current value of 0 will leave the basis. In this case, a degenerate bfs existed before and will result after the pivot. If more than one odd cell in the loop equals \( \theta \), you may arbitrarily choose one of these odd cells to leave the basis; again, a degenerate bfs will result.

We illustrate the pivoting procedure on the Powerco example. When the northwest corner method is applied to the Powerco example, the bfs in Table 33 is found. For this bfs, the basic variables are \( x_{11} = 35, x_{21} = 10, x_{22} = 20, x_{23} = 20, x_{33} = 10, \) and \( x_{34} = 30 \).

Suppose we want to find the bfs that would result if \( x_{34} \) were entered into the basis. The loop involving \( x_{14} \) and some of the basic variables is

\[
\begin{align*}
&\text{E} & \text{O} & \text{E} & \text{O} & \text{E} & \text{O} \\
& (1, 4) & (3, 4) & (3, 3) & (2, 3) & (2, 1) & (1, 1)
\end{align*}
\]

In this loop, (1, 4), (3, 3), and (2, 1) are the even cells, and (1, 1), (3, 4), and (2, 3) are the odd cells. The odd cell with the smallest value is \( x_{23} = 20 \). Thus, after the pivot, \( x_{23} \) will have left the basis. We now add 20 to each of the even cells and subtract 20 from each of the odd cells. The bfs in Table 34 results. Because each row and column has as many +20s as −20s, the new solution will satisfy each supply and demand constraint. By choosing the smallest odd variable (\( x_{23} \)) to leave the basis, we have ensured that all variables will remain non-negative. Thus, the new solution is feasible. There is no loop involving the cells (1, 1), (1, 4), (2, 1), (2, 2), (3, 3), and (3, 4), so the new solution is a bfs. After the pivot, the new bfs is \( x_{11} = 15, x_{14} = 20, x_{21} = 30, x_{22} = 20, x_{33} = 30, \) and \( x_{34} = 10, \) and all other variables equal 0.

**Table 33**

Northwest Corner Basic Feasible Solution for Powerco

<table>
<thead>
<tr>
<th></th>
<th>35</th>
<th></th>
<th></th>
<th></th>
<th>35</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>20</td>
<td>20</td>
<td></td>
<td></td>
<td>50</td>
</tr>
<tr>
<td>45</td>
<td>10</td>
<td>30</td>
<td></td>
<td></td>
<td>40</td>
</tr>
</tbody>
</table>

**Table 34**

New Basic Feasible Solution After \( x_{14} \) Is Pivoted into Basis

<table>
<thead>
<tr>
<th></th>
<th>35 − 20</th>
<th></th>
<th></th>
<th>0 + 20</th>
<th>35</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 + 20</td>
<td>20</td>
<td>20 − 20 (nonbasic)</td>
<td></td>
<td>50</td>
<td></td>
</tr>
<tr>
<td>45</td>
<td>10 + 20</td>
<td>30 − 20</td>
<td></td>
<td>40</td>
<td></td>
</tr>
</tbody>
</table>

7.3 The Transportation Simpler Method
The preceding illustration of the pivoting procedure makes it clear that each pivot in a transportation problem involves only additions and subtractions. Using this fact, we can show that if all the supplies and demands for a transportation problem are integers, then the transportation problem will have an optimal solution in which all the variables are integers. Begin by observing that, by the northwest corner method, we can find a bfs in which each variable is an integer. Each pivot involves only additions and subtractions, so each bfs obtained by performing the simplex algorithm (including the optimal solution) will assign all variables integer values. The fact that a transportation problem with integer supplies and demands has an optimal integer solution is useful, because it ensures that we need not worry about whether the Divisibility Assumption is justified.

**Pricing Out Nonbasic Variables (Based on Chapter 6)**

To complete our discussion of the transportation simplex, we now show how to compute row \( j \) for any bfs. From Section 6.2, we know that for a bfs in which the set of basic variables is BV, the coefficient of the variable \( x_{ij} \) (call it \( z_{ij} \)) in the tableau's \( i \)-row is given by

\[
z_{ij} = c_{ij} - c_{B}^{T}B^{-1}A_{ij}
\]

where \( c_{ij} \) is the objective function coefficient for \( x_{ij} \) and \( c_{B}^{T} \) is the column for \( x_{ij} \) in the original LP (we are assuming that the first supply constraint has been dropped).

Because we are solving a minimization problem, the current bfs will be optimal if all the \( z_{ij} \) are \text{nonnegative} \( z_{ij} \) otherwise, we enter into the basis the variable with the most \text{negative} \( z_{ij} \).

After determining \( c_{BV}B^{-1} \), we can easily determine \( z_{ij} \). Because the first constraint has been dropped, \( c_{BV}B^{-1} \) will have \( m + n - 1 \) elements. We write

\[
c_{BV}B^{-1} = [u_{2} \quad u_{3} \quad \ldots \quad u_{m} \quad v_{1} \quad v_{2} \quad \ldots \quad v_{n}]
\]

where \( u_{2}, u_{3}, \ldots, u_{m} \) are the elements of \( c_{BV}B^{-1} \) corresponding to the \( m - 1 \) supply constraints, and \( v_{1}, v_{2}, \ldots, v_{n} \) are the elements of \( c_{BV}B^{-1} \) corresponding to the \( n \) demand constraints.

To determine \( c_{BV}B^{-1} \), we use the fact that in any tableau, each basic variable \( x_{ij} \) must have \( z_{ij} = 0 \). Thus, for each of the \( m + n - 1 \) variables in BV,

\[
c_{ij} - c_{B}^{T}B^{-1}A_{ij} = 0 \tag{4}
\]

For a transportation problem, the equations in (4) are very easy to solve. To illustrate the solution of (4), we find \( c_{BV}B^{-1} \) for (5), by applying the northwest corner method bfs to the Powerco problem.

<table>
<thead>
<tr>
<th></th>
<th>35</th>
<th>8</th>
<th>6</th>
<th>10</th>
<th>9</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>35</td>
<td>9</td>
<td>12</td>
<td>13</td>
<td>7</td>
<td>50</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>20</td>
<td>20</td>
<td>16</td>
<td>30</td>
<td>5</td>
<td>40</td>
</tr>
<tr>
<td>45</td>
<td>20</td>
<td>10</td>
<td>30</td>
<td>5</td>
<td>40</td>
<td>384</td>
</tr>
</tbody>
</table>
(This book's formula for reduced costs is the negative of the one we have been using hence I've included the negative signs for our purposes.)

For each basic variable $x_i$, if we define $y_i = 0$, we see that (d) reduces to $x_i + y_i = c_i$, for all basic variables. Thus, $y_i = c_i$ for all basic variables.

For this bfs, BV = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}. Applying (d) we obtain
For (5), we find \( \mathbf{c} \mathbf{B}^{-1} \) by solving

\[
\begin{align*}
  u_1 &= 0 \\
  u_1 + v_1 &= 8 \\
  u_2 + v_1 &= 9 \\
  u_2 + v_2 &= 12 \\
  u_2 + v_3 &= 13 \\
  u_3 + v_2 &= 16 \\
  u_3 + v_4 &= 5 \\
\end{align*}
\]

From (7), \( v_1 = 8 \). From (8), \( u_2 = 1 \). Then (9) yields \( v_2 = 11 \), and (10) yields \( v_3 = 12 \).

From (11), \( u_3 = 4 \). Finally, (12) yields \( v_4 = 1 \). For each nonbasic variable, we now compute \( \bar{c}_y = \mathbf{c}_y - \mathbf{u}^T \mathbf{v} \). We obtain

\[
\begin{align*}
  -\bar{c}_{12} &= 0 + 11 - 6 = 5 \\
  -\bar{c}_{14} &= 0 + 1 - 9 = -8 \\
  -\bar{c}_{31} &= 4 + 8 - 14 = -2 \\
\end{align*}
\]

Because \( \bar{c}_{32} \) is the most positive \( \bar{c}_{ij} \), we would next enter \( x_{32} \) into the basis. Each unit of \( x_{32} \) that is entered into the basis will decrease Powerco's cost by $6.

\[\text{How to Determine the Entering Nonbasic Variable (Based on Chapter 5)}\]

For readers who have not covered Chapter 6, we now discuss how to determine whether a bfs is optimal, and, if it is not, how to determine which nonbasic variable should enter the basis. Let \( -u_i \) \((i = 1, 2, \ldots, m)\) be the shadow price of the \( i \)th supply constraint, and let \(-v_j\)
\((j = 1, 2, \ldots, n)\) be the shadow price of the \( j \)th demand constraint. We assume that the first supply constraint has been dropped, so we may set \(-u_1 = 0\). From the definition of shadow price, if we were to increase the right-hand side of the \( i \)th supply and \( j \)th demand constraint by 1, the optimal \( z \)-value would decrease by \(-u_i - v_j\). Equivalently, if we were to decrease the right-hand side of the \( i \)th supply and \( j \)th demand constraint by 1, the optimal \( z \)-value would increase by \(-u_i - v_j\). Now suppose \( x_y \) is a nonbasic variable. Should we enter \( x_y \) into the basis? Observe that if we increase \( x_y \) by 1, costs directly increase by \( c_y \). Also, increasing \( x_y \) by 1 means that one less unit will be shipped from supply point \( i \) and one less unit will be shipped to demand point \( j \). This is equivalent to reducing the right-hand sides of the \( i \)th supply constraint and \( j \)th demand constraint by 1. This will increase \( z \) by \(-u_i - v_j\). Thus, increasing \( x_y \) by 1 will increase \( z \) by a total of \( c_y - u_i - v_j \). So if \( c_y - u_i - v_j \geq 0 \) (or \( u_i + v_j - c_y \leq 0 \)) for all nonbasic variables, the current bfs will be optimal. If, however, a nonbasic variable \( x_y \) has \( c_y - u_i - v_j < 0 \) (or \( u_i + v_j - c_y > 0 \)), then \( z \) can be decreased by \( u_i + v_j - c_y \) per unit of \( x_y \) by entering \( x_y \) into the basis. Thus, we may conclude that if \( u_i + v_j - c_y \leq 0 \) for all nonbasic variables, then the current bfs is optimal. Otherwise, the nonbasic variable with the most positive value of \( u_i + v_j - c_y \) should enter the basis. How do we find the \( u_i \)'s and \( v_j \)'s? The coefficient of a nonbasic variable \( x_y \) in row 0 of any tableau is the amount by which a unit increase in \( x_y \) will decrease \( z \), so we can conclude that the coefficient of any nonbasic variable (and, it turns out, any basic variable) in row 0 is \( u_i + v_j - c_y \). So we may solve for the \( u_i \)'s and \( v_j \)'s by solving the following system of equations: \( u_i = 0 \) and \( u_i + v_j - c_y = 0 \) for all basic variables.

To illustrate the previous discussion, consider the bfs for the Powerco problem shown in (5).
We find the $u_i$'s and $v_j$'s by solving

\begin{align*}
    u_1 &= 0 \\
    u_1 + v_1 &= 8 \\
    u_2 + v_1 &= 9 \\
    u_2 + v_2 &= 12 \\
    u_2 + v_3 &= 13 \\
    u_3 + v_3 &= 16 \\
    u_3 + v_4 &= 5
\end{align*}

From (7), $v_1 = 8$. From (8), $u_2 = 1$. Then (9) yields $v_2 = 11$, and (10) yields $v_3 = 12$. From (11), $u_3 = 4$. Finally, (12) yields $v_4 = 1$. For each nonbasic variable, we now compute $\bar{c}_{ij} = u_i + v_j - c_{ij}$. We obtain

\begin{align*}
    \bar{c}_{12} &= 0 + 11 - 6 = 5 \\
    \bar{c}_{13} &= 0 + 12 - 10 = 2 \\
    \bar{c}_{14} &= 0 + 1 - 9 = -8 \\
    \bar{c}_{24} &= 1 + 1 - 7 = -5 \\
    \bar{c}_{31} &= 4 + 8 - 14 = -2 \\
    \bar{c}_{32} &= 4 + 11 - 9 = 6
\end{align*}

Because $\bar{c}_{32}$ is the most positive $\bar{c}_{ij}$, we would next enter $x_{32}$ into the basis. Each unit of $x_{32}$ that is entered into the basis will decrease Powerco's cost by $6$.

We can now summarize the procedure for using the transportation simplex to solve a transportation (min) problem.

**Summary and Illustration of the Transportation Simplex Method**

**Step 1** If the problem is unbalanced, balance it.

**Step 2** Use one of the methods described in Section 7.2 to find a bfs.

**Step 3** Use the fact that $u_1 = 0$ and $u_i + v_j = c_{ij}$ for all basic variables to find the $[u_1 \ u_2 \ \ldots \ u_m \ v_1 \ v_2 \ \ldots \ v_n]$ for the current bfs.

**Step 4** If $\bar{c}_{ij} - u_i - v_j \geq 0$ for all nonbasic variables, then the current bfs is optimal. If this is not the case, then we enter the variable with the most negative $\bar{c}_{ij} - u_i - v_j$ into the basis using the pivoting procedure. This yields a new bfs.

**Step 5** Using the new bfs, return to steps 3 and 4.

For a maximization problem, proceed as stated, but replace step 4 by step 4'.

**Step 4'** If $u_i + v_j - c_{ij} \geq 0$ for all nonbasic variables, then the current bfs is optimal. Otherwise, enter the variable with the most negative $u_i + v_j - c_{ij}$ into the basis using the pivoting procedure described earlier.
We illustrate the procedure for solving a transportation problem by solving the \( P \)-problem. We begin with the bfs (5). We have already determined that \( x_{32} \) should enter the basis. As shown in Table 35, the loop involving \( x_{32} \) and some of the basic variables is \((3, 2) \rightarrow (3, 3) \rightarrow (2, 3) \rightarrow (2, 2)\). The odd cells in this loop are \((3, 3)\) and \((2, 2)\). Because \( x_{33} = 10 \) and \( x_{22} = 20 \), the pivot will decrease the value of \( x_{33} \) and \( x_{22} \) by 10 and increase the value of \( x_{32} \) and \( x_{23} \) by 10. The resulting bfs is shown in Table 36. The \( u_i \)'s and \( v_j \)'s for the new bfs were obtained by solving

\[
\begin{align*}
  u_1 & = 0 & u_2 + v_3 & = 13 \\
  u_2 + v_2 & = 12 & u_2 + v_1 & = 9 \\
  u_3 + v_4 & = 5 & u_3 + v_2 & = 9 \\
  u_1 + v_1 & = 8 
\end{align*}
\]

In computing \( \bar{c}_{ij} = c_{ij} - u_i - v_j \) for each nonbasic variable, we find that \( \bar{c}_{12} = 5 \), \( \bar{c}_{24} = -1 \), and \( \bar{c}_{13} = -2 \) are the only negative \( \bar{c}_{ij} \)'s. Thus, we next enter \( x_{12} \) into the basis. The loop involving \( x_{12} \) and some of the basic variables is \((1, 2) \rightarrow (2, 2) \rightarrow (2, 1) \rightarrow (1, 1)\). The odd cells are \((2, 2)\) and \((1, 1)\). Because \( x_{22} = 10 \) is the smallest entry in an odd cell, we decrease \( x_{22} \) and \( x_{11} \) by 10 and increase \( x_{12} \) and \( x_{21} \) by 10. The resulting bfs is shown in Table 37. For this bfs, the \( u_i \)'s and \( v_j \)'s were determined by solving

\[
\begin{align*}
  u_1 & = 0 & u_1 + v_2 & = 6 \\
  u_2 + v_1 & = 9 & u_3 + v_2 & = 9 \\
  u_1 + v_1 & = 8 & u_3 + v_4 & = 5 \\
  u_2 + v_3 & = 13 
\end{align*}
\]

In computing \( \bar{c}_{ij} \) for each nonbasic variable, we find that the only negative \( \bar{c}_{ij} \) is \( \bar{c}_{13} = -2 \). Thus, \( x_{13} \) enters the basis. The loop involving \( x_{13} \) and some of the basic variables is

![Table 35](image)

![Table 36](image)
the Powerco variables. Because increase 1 and \( v_j \)’s

\( \tilde{c}_{12} = 5 \),

as is. The odd 1, we do

shown in

is \( \tilde{c}_{13} = 6 \).

(1, 3)–(2, 3)–(2, 1)–(1, 1). The odd cells are \( x_{23} \) and \( x_{11} \). Because \( x_{11} = 25 \) is the smallest entry in an odd cell, we decrease \( x_{23} \) and \( x_{11} \) by 25 and increase \( x_{13} \) and \( x_{21} \) by 25. The resulting bfs is shown in Table 38. For this bfs, the \( u_j \)’s and \( v_j \)’s were obtained by solving

\[
\begin{align*}
  u_1 &= 0 \\
  u_2 + v_3 &= 13 \\
  u_2 + v_1 &= 9 \\
  u_3 + v_4 &= 5 \\
  u_3 + v_2 &= 9 \\
  u_1 + v_2 &= 6
\end{align*}
\]

The reader should check that for this bfs, all \( \tilde{c}_{ij} \leq 0 \), so an optimal solution has been obtained. Thus, the optimal solution to the Powerco problem is \( x_{12} = 10, x_{13} = 25, x_{21} = 45, x_{23} = 5, x_{32} = 10, x_{34} = 30 \), and

\[
z = 6(10) + 10(25) + 9(45) + 13(5) + 9(10) + 5(30) = 1,020
\]

**PROBLEMS**

**Group A**

Use the transportation simplex to solve Problems 1–8 in Section 7.1. Begin with the bfs found in Section 7.2.