In this chapter, we start with a linear programming problem, called the primal, and introduce another linear programming problem, called the dual. Duality theory deals with the relation between these two problems and uncovers the deeper structure of linear programming. It is a powerful theoretical tool that has numerous applications, provides new geometric insights, and leads to another algorithm for linear programming (the dual simplex method).

4.1 Motivation

Duality theory can be motivated as an outgrowth of the Lagrange multiplier method, often used in calculus to minimize a function subject to equality constraints. For example, in order to solve the problem

\[
\begin{align*}
\text{minimize} & \quad x^2 + y^2 \\
\text{subject to} & \quad x + y = 1,
\end{align*}
\]

we introduce a Lagrange multiplier \( p \) and form the Lagrangean \( L(x, y, p) \) defined by

\[
L(x, y, p) = x^2 + y^2 + p(1 - x - y).
\]

While keeping \( p \) fixed, we minimize the Lagrangean over all \( x \) and \( y \), subject to no constraints, which can be done by setting \( \partial L/\partial x \) and \( \partial L/\partial y \) to zero. The optimal solution to this unconstrained problem is

\[
x = y = \frac{p}{2},
\]

and depends on \( p \). The constraint \( x + y = 1 \) gives us the additional relation \( p = 1 \), and the optimal solution to the original problem is \( x = y = 1/2 \).

The main idea in the above example is the following. Instead of enforcing the hard constraint \( x + y = 1 \), we allow it to be violated and associate a Lagrange multiplier, or price, \( p \) with the amount \( 1 - x - y \) by which it is violated. This leads to the unconstrained minimization of \( x^2 + y^2 + p(1 - x - y) \). When the price is properly chosen (\( p = 1 \), in our example), the optimal solution to the constrained problem is also optimal for the unconstrained problem. In particular, under that specific value of \( p \), the presence or absence of the hard constraint does not affect the optimal cost.

The situation in linear programming is similar: we associate a price variable with each constraint and start searching for prices under which the presence or absence of the constraints does not affect the optimal cost. It turns out that the right prices can be found by solving a new linear programming problem, called the dual of the original. We now motivate the form of the dual problem.

Consider the standard form problem

\[
\begin{align*}
\text{minimize} & \quad c'x \\
\text{subject to} & \quad Ax = b \\
& \quad x \geq 0,
\end{align*}
\]

which we call the primal problem, and let \( x^* \) be an optimal solution, assumed to exist. We introduce a relaxed problem in which the constraint \( Ax = b \) is replaced by a penalty \( p'(b - Ax) \), where \( p \) is a price vector of the same dimension as \( b \). We are then faced with the problem

\[
\begin{align*}
\text{minimize} & \quad c'x + p'(b - Ax) \\
\text{subject to} & \quad x \geq 0.
\end{align*}
\]

Let \( g(p) \) be the optimal cost for the relaxed problem, as a function of the price vector \( p \). The relaxed problem allows for more options than those present in the primal problem, and we expect \( g(p) \) to be no larger than the optimal primal cost \( c'x^* \). Indeed,

\[
g(p) = \min_{x \geq 0} \left[ c'x + p'(b - Ax) \right] \leq c'x^* + p'(b - Ax^*) = c'x^*,
\]

where the last inequality follows from the fact that \( x^* \) is a feasible solution to the primal problem, and satisfies \( Ax^* = b \). Thus, each \( p \) leads to a lower bound \( g(p) \) for the optimal cost \( c'x^* \). The problem

\[
\begin{align*}
\text{maximize} & \quad g(p) \\
\text{subject to} & \quad \text{no constraints}
\end{align*}
\]

can be then interpreted as a search for the tightest possible lower bound of this type, and is known as the dual problem. The main result in duality theory asserts that the optimal cost in the dual problem is equal to the optimal cost \( c'x^* \) in the primal. In other words, when the prices are chosen according to an optimal solution for the dual problem, the option of violating the constraints \( Ax = b \) is of no value.

Using the definition of \( g(p) \), we have

\[
g(p) = \min_{x \geq 0} \left[ c'x + p'(b - Ax) \right] = p'b + \min_{x \geq 0} (c' - p'A)x.
\]

Note that

\[
\min_{x \geq 0} (c' - p'A)x = \begin{cases} 
0, & \text{if } c' - p'A \geq 0', \\
-\infty, & \text{otherwise}.
\end{cases}
\]

In maximizing \( g(p) \), we only need to consider those values of \( p \) for which \( g(p) \) is not equal to \( -\infty \). We therefore conclude that the dual problem is the same as the linear programming problem

\[
\begin{align*}
\text{maximize} & \quad p'b \\
\text{subject to} & \quad p'A \leq c'.
\end{align*}
\]
In the preceding example, we started with the equality constraint $Ax = b$ and we ended up with no constraints on the sign of the price vector $p$. If the primal problem had instead inequality constraints of the form $Ax \geq b$, they could be replaced by $Ax - s = b, s \geq 0$. The equality constraint can be written in the form

$$\begin{bmatrix} A & -I \end{bmatrix} \begin{bmatrix} x \\ s \end{bmatrix} = 0,$$

which leads to the dual constraints

$$p'[A | -I] \leq [c' | 0'],$$

or, equivalently,

$$p'A \leq c', \quad p \geq 0.$$

Also, if the vector $x$ is free rather than sign-constrained, we use the fact

$$\min_{x} (c' - p'A)x = \begin{cases} 0, & \text{if } c' - p'A = 0' \\ -\infty, & \text{otherwise}, \end{cases}$$

to end up with the constraint $p'A = c'$ in the dual problem. These considerations motivate the general form of the dual problem which we introduce in the next section.

In summary, the construction of the dual of a primal minimization problem can be viewed as follows. We have a vector of parameters (dual variables) $p$, and for every $p$ we have a method for obtaining a lower bound on the optimal primal cost. The dual problem is a maximization problem that looks for the tightest such lower bound. For some vectors $p$, the corresponding lower bound is equal to $-\infty$, and does not carry any useful information. Thus, we only need to maximize over those $p$ that lead to nontrivial lower bounds, and this is what gives rise to the dual constraints.

### 4.2 The dual problem

Let $A$ be a matrix with rows $a'_i$ and columns $A_j$. Given a primal problem with the structure shown on the left, its dual is defined to be the maximization problem shown on the right:

**minimize** \[ c'x \]
**subject to** \[ a'_i x \geq b_i, \quad i \in M_1, \]
\[ a'_i x \leq b_i, \quad i \in M_2, \]
\[ a'_i x = b_i, \quad i \in M_3, \]
\[ x_j \geq 0, \quad j \in N_1, \]
\[ x_j \leq 0, \quad j \in N_2, \]
\[ x_j \text{ free}, \quad j \in N_3, \]

**maximize** \[ pb \]
**subject to** \[ p_i \geq 0, \quad i \in M_1, \]
\[ p_i \leq 0, \quad i \in M_2, \]
\[ p_i \text{ free}, \quad i \in M_3, \]
\[ p'A_j \leq c_j, \quad j \in N_1, \]
\[ p'A_j \geq c_j, \quad j \in N_2, \]
\[ p'A_j = c_j, \quad j \in N_3. \]

Notice that for each constraint in the primal (other than the sign constraints), we introduce a variable in the dual problem; for each variable in the primal, we introduce a constraint in the dual. Depending on whether the primal constraint is an equality or inequality constraint, the corresponding dual variable is either free or sign-constrained, respectively. In addition, depending on whether a variable in the primal problem is free or sign-constrained, we have an equality or inequality constraint, respectively, in the dual problem. We summarize these relations in Table 4.1.

<table>
<thead>
<tr>
<th>PRIMAL</th>
<th>minimize</th>
<th>maximize</th>
<th>DUAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>constraints</td>
<td>$\geq b_i$</td>
<td>$\geq 0$</td>
<td>variables</td>
</tr>
<tr>
<td>constraints</td>
<td>$\leq b_i$</td>
<td>$\leq 0$</td>
<td></td>
</tr>
<tr>
<td>variables</td>
<td>$= b_i$</td>
<td>free</td>
<td></td>
</tr>
<tr>
<td>variables</td>
<td>$\geq 0$</td>
<td>$\leq c_j$</td>
<td></td>
</tr>
<tr>
<td>variables</td>
<td>$\leq 0$</td>
<td>$\geq c_j$</td>
<td></td>
</tr>
<tr>
<td>variables</td>
<td>free</td>
<td>$= c_j$</td>
<td></td>
</tr>
</tbody>
</table>

**Table 4.1**: Relation between primal and dual variables and constraints.

If we start with a maximization problem, we can always convert it into an equivalent minimization problem, and then form its dual according to the rules we have described. However, to avoid confusion, we will adhere to the convention that the primal is a minimization problem, and its dual is a maximization problem. Finally, we will keep referring to the objective function in the dual problem as a "cost" that is being maximized.

A problem and its dual can be stated more compactly, in matrix notation, if a particular form is assumed for the primal. We have, for example, the following pairs of primal and dual problems:

**minimize** \[ c'x \]
**subject to** \[ Ax = b, \]
\[ x \geq 0, \]

**maximize** \[ p'b \]
**subject to** \[ p'A \leq c', \]

and

**minimize** \[ c'x \]
**subject to** \[ Ax \geq b, \]

**maximize** \[ p'b \]
**subject to** \[ p'A = c', \]
\[ p \geq 0. \]

**Example 4.1** Consider the primal problem shown on the left and its dual shown
on the right:

minimize \( x_1 + 2x_2 + 3x_3 \) 
subject to \(-x_1 + 3x_2 = 5\)
\(2x_1 - x_2 + 3x_3 \geq 6\)
\(x_3 \leq 4\)
\(x_1 \geq 0\)
\(x_2 \leq 0\)
\(x_3 \text{ free}\),

maximize \( 5p_1 + 6p_2 + 4p_3 \)
subject to \(p_1 \text{ free}\)
\(p_2 \geq 0\)
\(p_3 \leq 0\)
\(-p_1 + 2p_2 \leq 1\)
\(3p_1 - p_2 \geq 2\)
\(3p_2 + p_3 = 3\).

We transform the dual into an equivalent minimization problem, rename the variables from \(p_1,p_2,p_3\) to \(x_1,x_2,x_3\), and multiply the three last constraints by \(-1\). The resulting problem is shown on the left. Then, on the right, we show its dual:

minimize \(-5x_1 - 6x_2 - 4x_3\) 
subject to \(x_1 \text{ free}\)
\(x_2 \geq 0\)
\(x_3 \leq 0\)
\(x_1 - 2x_2 \geq -1\)
\(-3x_1 + x_2 \leq -2\)
\(-3x_2 - x_3 = -3\),

maximize \(-p_1 - 2p_2 - 3p_3\)
subject to \(p_1 \geq 0\)
\(p_2 \leq 0\)
\(-p_3 \geq -4\)

We observe that the latter problem is equivalent to the primal problem we started with. (The first three constraints in the latter problem are the same as the first three constraints in the original problem, multiplied by \(-1\). Also, if the maximization in the latter problem is changed to a minimization, by multiplying the objective function by \(-1\), we obtain the cost function in the original problem.)

The first primal problem considered in Example 4.1 had all of the ingredients of a general linear programming problem. This suggests that the conclusion reached at the end of the example should hold in general. Indeed, we have the following result. Its proof needs nothing more than the steps followed in Example 4.1, with abstract symbols replacing specific numbers, and will therefore be omitted.

**Theorem 4.1** If we transform the dual into an equivalent minimization problem and then form its dual, we obtain a problem equivalent to the original problem.

A compact statement that is often used to describe Theorem 4.1 is that “the dual of the dual is the primal.”

Any linear programming problem can be manipulated into one of several equivalent forms, for example, by introducing slack variables or by using the difference of two nonnegative variables to replace a single free variable. Each equivalent form leads to a somewhat different form for the dual problem. Nevertheless, the examples that follow indicate that the duals of equivalent problems are equivalent.

**Example 4.2** Consider the primal problem shown on the left and its dual shown on the right:

minimize \( c'x \) 
subject to \( Ax \geq b \)
\( x \text{ free}\),

maximize \( p'b \) 
subject to \( p \geq 0 \)
\( p'A = c' \).

We transform the primal problem by introducing surplus variables and then obtain its dual:

minimize \( c'x + 0's \) 
subject to \( Ax - s = b \)
\( x \text{ free}\)
\( s \geq 0 \),

maximize \( p'b \) 
subject to \( p \geq 0 \)
\( p'A = c' \)
\( -p \leq 0 \).

Alternatively, if we take the original primal problem and replace \( x \) by sign-constrained variables, we obtain the following pair of problems:

minimize \( c'x^+ - c'x^- \) 
subject to \( Ax^+ - Ax^- \geq b \)
\( x^+ \geq 0 \)
\( x^- \geq 0 \),

maximize \( p'b \) 
subject to \( p \geq 0 \)
\( p'A \leq c' \)
\( -p \leq -c' \).

Note that we have three equivalent forms of the primal. We observe that the constraint \( p \geq 0 \) is equivalent to the constraint \(-p \leq 0 \). Furthermore, the constraint \( p'A = c' \) is equivalent to the two constraints \( p'A \leq c' \) and \(-p'A \leq -c' \). Thus, the duals of the three variants of the primal problem are also equivalent.

The next example is in the same spirit and examines the effect of removing redundant equality constraints in a standard form problem.

**Example 4.3** Consider a standard form problem, assumed feasible, and its dual:

minimize \( c'x \) 
subject to \( Ax = b \)
\( x \geq 0 \),

maximize \( p'b \) 
subject to \( p'A \leq c' \).

Let \( a_1', \ldots, a_m' \) be the rows of \( A \) and suppose that \( a_m' = \sum_{i=1}^{m-1} \gamma_i a_i \) for some scalars \( \gamma_1, \ldots, \gamma_{m-1} \). In particular, the last equality constraint is redundant and can be eliminated. By considering an arbitrary feasible solution \( x \), we obtain

\[
b_m = a_m'x = \sum_{i=1}^{m-1} \gamma_i a_i'x = \sum_{i=1}^{m-1} \gamma_i b_i.
\] (4.1)

Note that the dual constraints are of the form \( \sum_{i=1}^{m-1} p_i a_i' \leq c' \) and can be rewritten as

\[
\sum_{i=1}^{m-1} (p_i + \gamma_i p_m) a_i' \leq c'.
\]

Furthermore, using Eq. (4.1), the dual cost \( \sum_{i=1}^{m-1} p_i b_i \) is equal to

\[
\sum_{i=1}^{m-1} (p_i + \gamma_i p_m) b_i.
\]
If we now let \( q_i = p_i + \gamma_i p_m \), we see that the dual problem is equivalent to

\[
\begin{align*}
\text{maximize} & \quad \sum_{i=1}^{m-1} q_i b_i \\
\text{subject to} & \quad \sum_{i=1}^{m-1} q_i a_i' \leq c'.
\end{align*}
\]

We observe that this is the exact same dual that we would have obtained if we had eliminated the last (and redundant) constraint in the primal problem, before forming the dual.

The conclusions of the preceding two examples are summarized and generalized by the following result.

**Theorem 4.2** Suppose that we have transformed a linear programming problem \( \Pi_i \) to another linear programming problem \( \Pi_2 \), by a sequence of transformations of the following types:

(a) Replace a free variable with the difference of two nonnegative variables.

(b) Replace an inequality constraint by an equality constraint involving a nonnegative slack variable.

(c) If some row of the matrix \( A \) in a feasible standard form problem is a linear combination of the other rows, eliminate the corresponding equality constraint.

Then, the duals of \( \Pi_1 \) and \( \Pi_2 \) are equivalent, i.e., they are either both infeasible, or they have the same optimal cost.

The proof of Theorem 4.2 involves a combination of the various steps in Examples 4.2 and 4.3, and is left to the reader.

### 4.3 The duality theorem

We saw in Section 4.1 that for problems in standard form, the cost \( g(p) \) of any dual solution provides a lower bound for the optimal cost. We now show that this property is true in general.

**Theorem 4.3 (Weak duality)** If \( x \) is a feasible solution to the primal problem and \( p \) is a feasible solution to the dual problem, then

\[ p'b \leq c'x \]

**Proof.** For any vectors \( x \) and \( p \), we define

\[
\begin{align*}
u_i &= p_i(a'_ix - b_i), \\
v_j &= (c_j - p'A_j)x_j.
\end{align*}
\]

Suppose that \( x \) and \( p \) are primal and dual feasible, respectively. The definition of the dual problem requires the sign of \( p_i \) to be the same as the sign of \( a'_ix - b_i \), and the sign of \( c_j - p'A_j \) to be the same as the sign of \( x_j \). Thus, primal and dual feasibility imply that

\[ u_i \geq 0, \quad \forall \ i, \]

and

\[ v_j \geq 0, \quad \forall \ j. \]

Notice that

\[ \sum_i u_i = p'Ax - p'b, \]

and

\[ \sum_j v_j = c'x - p'Ax. \]

We add these two equalities and use the nonnegativity of \( u_i, v_j \), to obtain

\[ 0 \leq \sum_i u_i + \sum_j v_j = c'x - p'b. \]

The weak duality theorem is not a deep result, yet it does provide some useful information about the relation between the primal and the dual. We have, for example, the following corollary.

**Corollary 4.1**

(a) If the optimal cost in the primal is \(-\infty\), then the dual problem must be infeasible.

(b) If the optimal cost in the dual is \(+\infty\), then the primal problem must be infeasible.

**Proof.** Suppose that the optimal cost in the primal problem is \(-\infty\) and that the dual problem has a feasible solution \( p \). By weak duality, \( p'b \leq c'x \) for every primal feasible \( x \). Taking the minimum over all primal feasible \( x \), we conclude that \( p'b \leq -\infty \). This is impossible and shows that the dual cannot have a feasible solution, thus establishing part (a). Part (b) follows by a symmetrical argument.

Another important corollary of the weak duality theorem is the following.
Corollary 4.2. Let \( x \) and \( p \) be feasible solutions to the primal and the dual, respectively, and suppose that \( p^T b = c^T x \). Then, \( x \) and \( p \) are optimal solutions to the primal and the dual, respectively.

Proof. Let \( x \) and \( p \) be as in the statement of the corollary. For every primal feasible solution \( y \), the weak duality theorem yields \( c^T x = p^T b \leq c^T y \), which proves that \( x \) is optimal. The proof of optimality of \( p \) is similar.

The next theorem is the central result on linear programming duality.

Theorem 4.4 (Strong duality). If a linear programming problem has an optimal solution, so does its dual, and the respective optimal costs are equal.

Proof. Consider the standard form problem

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad A x = b \\
& \quad x \geq 0.
\end{align*}
\]

Let us assume temporarily that the rows of \( A \) are linearly independent and that there exists an optimal solution. Let us apply the simplex method to this problem. As long as cycling is avoided, e.g., by using the lexicographic pivoting rule, the simplex method terminates with an optimal solution \( x \) and an optimal basis \( B \). Let \( x_B = B^{-1} b \) be the corresponding vector of basic variables. When the simplex method terminates, the reduced costs must be nonnegative and we obtain

\[
c^T - c'_B B^{-1} A \geq 0,
\]

where \( c'_B \) is the vector with the costs of the basic variables. Let us define a vector \( p \) by letting \( p' = c'_B B^{-1} \). We then have \( p^T A \leq c' \), which shows that \( p \) is a feasible solution to the dual problem

\[
\begin{align*}
\text{maximize} & \quad p^T b \\
\text{subject to} & \quad p^T A \leq c'.
\end{align*}
\]

In addition,

\[
p^T b = c'_B B^{-1} b = c'_B x_B = c^T x.
\]

It follows that \( p \) is an optimal solution to the dual (cf. Corollary 4.2), and the optimal dual cost is equal to the optimal primal cost.

If we are dealing with a general linear programming problem \( \Pi \), that has an optimal solution, we first transform it into an equivalent standard form problem \( \Pi_2 \), with the same optimal cost, and in which the rows of the matrix \( A \) are linearly independent. Let \( D_1 \) and \( D_2 \) be the duals of \( \Pi_1 \) and \( \Pi_2 \), respectively. By Theorem 4.2, the dual problems \( D_1 \) and \( D_2 \) have the same optimal cost. We have already proved that \( \Pi_2 \) and \( D_2 \) have the same optimal cost. It follows that \( \Pi_1 \) and \( D_1 \) have the same optimal cost (see Figure 4.1).

![Figure 4.1: Proof of the duality theorem for general linear programming problems.](image)

The preceding proof shows that an optimal solution to the dual problem is obtained as a byproduct of the simplex method as applied to a primal problem in standard form. It is based on the fact that the simplex method is guaranteed to terminate and this, in turn, depends on the existence of pivoting rules that prevent cycling. There is an alternative derivation of the duality theorem, which provides a geometric, algorithm-independent view of the subject, and which is developed in Section 4.7. At this point, we provide an illustration that conveys most of the content of the geometric proof.

Example 4.4 Consider a solid ball constrained to lie in a polyhedron defined by inequality constraints of the form \( a_i x \geq b_i \). If left under the influence of gravity, this ball reaches equilibrium at the lowest corner \( x^* \) of the polyhedron; see Figure 4.2. This corner is an optimal solution to the problem

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad a_i^T x \geq b_i, \quad \forall \ i,
\end{align*}
\]

where \( c \) is a vertical vector pointing upwards. At equilibrium, gravity is counterbalanced by the forces exerted on the ball by the "walls" of the polyhedron. The latter forces are normal to the walls, that is, they are aligned with the vectors \( a_i \). We conclude that \( c = \sum_i p_i a_i \), for some nonnegative coefficients \( p_i \); in particular,
the vector \( \mathbf{p} \) is a feasible solution to the dual problem

\[
\begin{align*}
\text{maximize} & \quad \mathbf{p}' \mathbf{b} \\
\text{subject to} & \quad \mathbf{p}' \mathbf{A} = \mathbf{c}' \\
& \quad \mathbf{p} \geq 0.
\end{align*}
\]

Given that forces can only be exerted by the walls that touch the ball, we must have \( p_i = 0 \), whenever \( a_i' \mathbf{x}^* > b_i \). Consequently, \( p_i (b_i - a_i' \mathbf{x}^*) = 0 \) for all \( i \). We therefore have \( \mathbf{p}' \mathbf{b} = \sum_i p_i b_i = \sum_i p_i a_i' \mathbf{x}^* = \mathbf{c}' \mathbf{x}^* \). It follows (Corollary 4.2) that \( \mathbf{p} \) is an optimal solution to the dual, and the optimal dual cost is equal to the optimal primal cost.

![Figure 4.2: A mechanical analogy of the duality theorem.](image)

Recall that in a linear programming problem, exactly one of the following three possibilities will occur:

(a) There is an optimal solution.

(b) The problem is "unbounded"; that is, the optimal cost is \(-\infty\) (for minimization problems), or \(+\infty\) (for maximization problems).

(c) The problem is infeasible.

This leads to nine possible combinations for the primal and the dual, which are shown in Table 4.2. By the strong duality theorem, if one problem has an optimal solution, so does the other. Furthermore, as discussed earlier, the weak duality theorem implies that if one problem is unbounded, the other must be infeasible. This allows us to mark some of the entries in Table 4.2 as "impossible."

<table>
<thead>
<tr>
<th></th>
<th>Finite optimum</th>
<th>Unbounded</th>
<th>Infeasible</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Finite optimum</strong></td>
<td>Possible</td>
<td>Impossible</td>
<td>Impossible</td>
</tr>
<tr>
<td><strong>Unbounded</strong></td>
<td>Impossible</td>
<td>Impossible</td>
<td>Possible</td>
</tr>
<tr>
<td><strong>Infeasible</strong></td>
<td>Impossible</td>
<td>Possible</td>
<td>Possible</td>
</tr>
</tbody>
</table>

Table 4.2: The different possibilities for the primal and the dual.

The case where both problems are infeasible can indeed occur, as shown by the following example.

**Example 4.5** Consider the infeasible primal

\[
\begin{align*}
\text{minimize} & \quad x_1 + 2x_2 \\
\text{subject to} & \quad x_1 + x_2 = 1 \\
& \quad 2x_1 + 2x_2 = 3.
\end{align*}
\]

Its dual is

\[
\begin{align*}
\text{maximize} & \quad p_1 + 3p_2 \\
\text{subject to} & \quad p_1 + 2p_2 = 1 \\
& \quad p_1 + 2p_2 = 2,
\end{align*}
\]

which is also infeasible.

There is another interesting relation between the primal and the dual which is known as Clark’s theorem (Clark, 1961). It asserts that unless both problems are infeasible, at least one of them must have an unbounded feasible set (Exercise 4.21).

**Complementary slackness**

An important relation between primal and dual optimal solutions is provided by the **complementary slackness** conditions, which we present next.

**Theorem 4.5** (Complementary slackness) Let \( \mathbf{x} \) and \( \mathbf{p} \) be feasible solutions to the primal and the dual problem, respectively. The vectors \( \mathbf{x} \) and \( \mathbf{p} \) are optimal solutions for the two respective problems if and only if:

\[
\begin{align*}
p_i (a_i' \mathbf{x} - b_i) &= 0, & \forall i, \\
(c_j - \mathbf{p}' \mathbf{A}_j) x_j &= 0, & \forall j.
\end{align*}
\]

**Proof.** In the proof of Theorem 4.3, we defined \( u_i = p_i (a_i' \mathbf{x} - b_i) \) and \( v_j = (c_j - \mathbf{p}' \mathbf{A}_j) x_j \), and noted that for \( \mathbf{x} \) primal feasible and \( \mathbf{p} \) dual feasible,
we have \( u_i \geq 0 \) and \( v_j \geq 0 \) for all \( i \) and \( j \). In addition, we showed that

\[
c'x - p' b = \sum_i u_i + \sum_j v_j.
\]

By the strong duality theorem, if \( x \) and \( p \) are optimal, then \( c'x = p'b \), which implies that \( u_i = v_j = 0 \) for all \( i, j \). Conversely, if \( u_i = v_j = 0 \) for all \( i, j \), then \( c'x = p'b \), and Corollary 4.2 implies that \( x \) and \( p \) are optimal.

The first complementary slackness condition is automatically satisfied by every feasible solution to a problem in standard form. If the primal problem is not in standard form and has a constraint like \( a_i'x \geq b_i \), the corresponding complementary slackness condition asserts that the dual variable \( p_i \) is zero unless the constraint is active. An intuitive explanation is that a constraint which is not active at an optimal solution can be removed from the problem without affecting the optimal cost, and there is no point in associating a nonzero price with such a constraint. Note also the analogy with Example 4.4, where "forces" were only exerted by the active constraints.

If the primal problem is in standard form and a nondegenerate optimal basic feasible solution is known, the complementary slackness conditions determine a unique solution to the dual problem. We illustrate this fact in the next example.

**Example 4.6** Consider a problem in standard form and its dual:

\[
\begin{align*}
\text{minimize} & \quad 13x_1 + 10x_2 + 6x_3 \\
\text{subject to} & \quad 5x_1 + x_2 + 3x_3 = 8 \\
& \quad 3x_1 + x_2 = 3 \\
& \quad x_1, x_2, x_3 \geq 0,
\end{align*}
\]

\[
\begin{align*}
\text{maximize} & \quad 8p_1 + 3p_2 \\
\text{subject to} & \quad 5p_1 + 3p_2 \leq 13 \\
& \quad p_1 + p_2 \leq 10 \\
& \quad 3p_1 \leq 6.
\end{align*}
\]

As will be verified shortly, the vector \( x^* = (1, 0, 1) \) is a nondegenerate optimal solution to the primal problem. Assuming this to be the case, we use the complementary slackness conditions to construct the optimal solution to the dual. The condition \( p_1(a_1'x^* - b_1) = 0 \) is automatically satisfied for each \( i \), since the primal is in standard form. The condition \( (c_j - p'A_j)x_j^* = 0 \) is clearly satisfied for \( j = 2 \), because \( x^*_j = 0 \). However, since \( x^*_1 > 0 \) and \( x^*_3 > 0 \), we obtain

\[
5p_1 + 3p_2 = 13,
\]

and

\[
3p_1 = 6,
\]

which we can solve to obtain \( p_1 = 2 \) and \( p_2 = 1 \). Note that this is a dual feasible solution whose cost is equal to 19, which is the same as the cost of \( x^* \). This verifies that \( x^* \) is indeed an optimal solution as claimed earlier.

We now generalize the above example. Suppose that \( x_j \) is a basic variable in a nondegenerate optimal basic feasible solution to a primal problem in standard form. Then, the complementary slackness condition

\[
(c_j - p'A_j)x_j = 0 \quad \text{yields} \quad p'A_j = c_j \quad \text{for every such} \ j.
\]

Since the basic columns \( A_j \) are linearly independent, we obtain a system of equations for \( p \) which has a unique solution, namely, \( p' = c'pB^{-1} \). A similar conclusion can also be drawn for problems not in standard form (Exercise 4.12). On the other hand, if we are given a degenerate optimal basic feasible solution to the primal, complementary slackness may be of very little help in determining an optimal solution to the dual problem (Exercise 4.17).

We finally mention that if the primal constraints are of the form

\[
Ax \geq b, \quad x \geq 0,
\]

and the primal problem has an optimal solution, then there exist optimal solutions to the primal and the dual which satisfy strict complementary slackness; that is, a variable in one problem is nonzero if and only if the corresponding constraint in the other problem is active (Exercise 4.20). This result has some interesting applications in discrete optimization, but these lie outside the scope of this book.

**A geometric view**

We now develop a geometric view that allows us to visualize pairs of primal and dual vectors without having to draw the dual feasible set.

We consider the primal problem

\[
\begin{align*}
\text{minimize} & \quad c'x \\
\text{subject to} & \quad a_i'x \geq b_i, \quad i = 1, \ldots, m,
\end{align*}
\]

where the dimension of \( x \) is equal to \( n \). We assume that the vectors \( a_i \) span \( \mathbb{R}^n \). The corresponding dual problem is

\[
\begin{align*}
\text{maximize} & \quad p'b \\
\text{subject to} & \quad \sum_{i=1}^m p_i a_i = c \\
& \quad p \geq 0.
\end{align*}
\]

Let \( I \) be a subset of \( \{1, \ldots, m\} \) of cardinality \( n \), such that the vectors \( a_i, i \in I \), are linearly independent. The system \( a_i'x = b_i, i \in I \), has a unique solution, denoted by \( x^i \), which is a basic solution to the primal problem (cf. Definition 2.9 in Section 2.2). We assume, that \( x^i \) is nondegenerate, that is, \( a_i'x^i \neq b_i \) for \( i \notin I \).

Let \( p \in \mathbb{R}^m \) be a dual vector (not necessarily dual feasible), and let us consider what is required for \( x^i \) and \( p \) to be optimal solutions to the primal and the dual problem, respectively. We need:

\[
\begin{align*}
(a) \quad & a_i'x^i \geq b_i, \quad \text{for all} \ i, \\
(b) \quad & p_i = 0, \quad \text{for all} \ i \notin I, \\
(c) \quad & \sum_{i=1}^m p_i a_i = c, \\
(d) \quad & p \geq 0,
\end{align*}
\]

(primal feasibility),

(complementary slackness),

(dual feasibility),

(dual feasibility).
Given the complementary slackness condition (b), condition (c) becomes
\[ \sum p_i a_i = c. \]
Since the vectors \( a_i \), \( i \in I \), are linearly independent, the latter equation has a unique solution that we denote by \( p_i \). In fact, it is readily seen that the vectors \( a_i \), \( i \in I \), form a basis for the dual problem (which is in standard form) and \( p_i \) is the associated basic solution. For the vector \( p_i \) to be dual feasible, we also need it to be nonnegative. We conclude that once the complementary slackness condition (b) is enforced, feasibility of

Figure 4.3: Consider a primal problem with two variables and five inequality constraints \( (n = 2, m = 5) \), and suppose that no two of the vectors \( a_i \) are collinear. Every two-element subset \( I \) of \( \{1, 2, 3, 4, 5\} \) determines basic solutions \( x^I \) and \( p^I \) of the primal and the dual, respectively.

If \( I = \{1, 2\} \), \( x^I \) is primal infeasible (point A) and \( p^I \) is dual infeasible, because \( c \) is not a nonnegative linear combination of the vectors \( a_1 \) and \( a_2 \).

If \( I = \{1, 3\} \), \( x^I \) is primal feasible (point B) and \( p^I \) is dual infeasible.

If \( I = \{1, 4\} \), \( x^I \) is primal feasible (point C) and \( p^I \) is dual feasible, because \( c \) can be expressed as a nonnegative linear combination of the vectors \( a_1 \) and \( a_4 \). In particular, \( x^I \) and \( p^I \) are optimal.

If \( I = \{1, 5\} \), \( x^I \) is primal infeasible (point D) and \( p^I \) is dual feasible.

4.4 Optimal dual variables as marginal costs

In this section, we elaborate on the interpretation of the dual variables as prices. This theme will be revisited, in more depth, in Chapter 5.

Consider the standard form problem
\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax = b \\
& \quad x \geq 0.
\end{align*}
\]

We assume that the rows of \( A \) are linearly independent and that there
is a nondegenerate basic feasible solution \( x^* \) which is optimal. Let \( B \) be the corresponding basis matrix and let \( x_B = B^{-1}b \) be the vector of basic variables, which is positive, by nondegeneracy. Let us now replace \( b \) by \( b + d \), where \( d \) is a small perturbation vector. Since \( B^{-1}b > 0 \), we also have \( B^{-1}(b + d) > 0 \), as long as \( d \) is small. This implies that the same basis leads to a basic feasible solution of the perturbed problem as well. Perturbing the right-hand side vector \( b \) has no effect on the reduced costs associated with this basis. By the optimality of \( x^* \) in the original problem, the vector of reduced costs \( c' - c'B^{-1}A \) is nonnegative and this establishes that the same basis is optimal for the perturbed problem as well. Thus, the optimal cost in the perturbed problem is

\[
c'_p B^{-1}(b + d) = p'(b + d),
\]

where \( p' = c'_p B^{-1} \) is an optimal solution to the dual problem. Therefore, a small change of \( d \) in the right-hand side vector \( b \) results in a change of \( p'd \) in the optimal cost. We conclude that each component \( p_i \) of the optimal dual vector can be interpreted as the marginal cost (or shadow price) per unit increase of the \( i \)th requirement \( b_i \).

We conclude with yet another interpretation of duality, for standard form problems. In order to develop some concrete intuition, we phrase our discussion in terms of the diet problem (Example 1.3 in Section 1.1). We interpret each vector \( A_j \) as the nutritional content of the \( j \)th available food, and view \( b \) as the nutritional content of an ideal food that we wish to synthesize. Let us interpret \( p_i \) as the “fair” price per unit of the \( i \)th nutrient. A unit of the \( j \)th food has a value of \( c_j \) at the food market, but it also has a value of \( p'A_j \) if priced at the nutrient market. Complementary slackness asserts that every food which is used (at a nonzero level) to synthesize the ideal food, should be consistently priced at the two markets. Thus, duality is concerned with two alternative ways of cost accounting. The value of the ideal food, as computed in the food market, is \( c'x^* \), where \( x^* \) is an optimal solution to the primal problem; the value of the ideal food, as computed in the nutrient market, is \( p'b \). The duality relation \( c'x^* = p'b \) states that when prices are chosen appropriately, the two accounting methods should give the same results.

### 4.5 Standard form problems and the dual simplex method

In this section, we concentrate on the case where the primal problem is in standard form. We develop the dual simplex method, which is an alternative to the simplex method of Chapter 3. We also comment on the relation between the basic feasible solutions to the primal and the dual, including a discussion of dual degeneracy.

In the proof of the strong duality theorem, we considered the simplex method applied to a primal problem in standard form and defined a dual vector \( p \) by letting \( p' = c'_p B^{-1} \). We then noted that the primal optimality condition \( c' - c'_p B^{-1}A \geq 0 \) is the same as the dual feasibility condition \( p'A \leq c' \). We can thus think of the simplex method as an algorithm that maintains primal feasibility and works towards dual feasibility. A method with this property is generally called a primal algorithm. An alternative is to start with a dual feasible solution and work towards primal feasibility. A method of this type is called a dual algorithm. In this section, we present a dual simplex method, implemented in terms of the full tableau. We argue that it does indeed solve the dual problem, and we show that it moves from one basic feasible solution of the dual problem to another. An alternative implementation that only keeps track of the matrix \( B^{-1} \), instead of the entire tableau, is called a revised dual simplex method (Exercise 4.23).

The dual simplex method

Let us consider a problem in standard form, under the usual assumption that the rows of the matrix \( A \) are linearly independent. Let \( B \) be a basis matrix, consisting of \( m \) linearly independent columns of \( A \), and consider the corresponding tableau

<table>
<thead>
<tr>
<th>(-c'_p B^{-1}b)</th>
<th>(c')</th>
</tr>
</thead>
<tbody>
<tr>
<td>(B^{-1}b)</td>
<td>(B^{-1}A)</td>
</tr>
</tbody>
</table>

or, in more detail,

<table>
<thead>
<tr>
<th>(-c'_B x_B)</th>
<th>(c_1)</th>
<th>\cdots</th>
<th>(c_n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_B(1))</td>
<td>(\vdots)</td>
<td>(B^{-1}A_1)</td>
<td>(\vdots)</td>
</tr>
<tr>
<td>(x_B(m))</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
</tr>
</tbody>
</table>

We do not require \( B^{-1}b \) to be nonnegative, which means that we have a basic, but not necessarily feasible solution to the primal problem. However, we assume that \( c \geq 0 \); equivalently, the vector \( p' = c'_p B^{-1} \) satisfies \( p'A \leq c' \), and we have a feasible solution to the dual problem. The cost of this dual feasible solution is \( p'b = c'_p B^{-1}b = c'_B x_B \), which is the negative of the entry at the upper left corner of the tableau. If the inequality \( B^{-1}b \geq 0 \) happens to hold, we also have a primal feasible solution with the same cost, and optimal solutions to both problems have been found. If the inequality \( B^{-1}b \geq 0 \) fails to hold, we perform a change of basis in a manner we describe next.