Then obtain the new current row, column, and entry by moving down one row and one column to the right, and go to step 3. (b) If \( a_{11} \) (the current entry) equals 0, then do a Type 3 ERO involving the current row and any row that contains a nonzero number in the current column. Use EROs to transform column 1 to
\[
\begin{bmatrix}
1 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]
Then obtain the new current row, column, and entry by moving down one row and one column to the right. Go to Step 3. (c) If there are no nonzero numbers in the first column, then obtain a new current column and entry by moving one column to the right. Then go to step 3.

**Step 3**  
(a) If the new current entry is nonzero, then use EROs to transform it to 1 and the rest of the current column's entries to 0. When finished, obtain the new current row, column, and entry. If this is impossible, then stop. Otherwise, repeat step 3. (b) If the current entry is 0, then do a Type 3 ERO with the current row and any row that contains a nonzero number in the current column. Then use EROs to transform that current entry to 1 and the rest of the current column's entries to 0. When finished, obtain the new current row, column, and entry. If this is impossible, then stop. Otherwise, repeat step 3. (c) If the current column has no nonzero numbers below the current row, then obtain the new current column and entry, and repeat step 3. If it is impossible, then stop.

This procedure may require "passing over" one or more columns without transforming them (see Problem 8).

**Step 4**  
Write down the system of equations \( A'x = b' \) that corresponds to the matrix \( A'|b' \) obtained when step 3 is completed. Then \( A'x = b' \) will have the same set of solutions as \( Ax = b \).

**Basic Variables and Solutions to Linear Equation Systems**

To describe the set of solutions to \( A'x = b' \) (and \( Ax = b \)), we need to define the concepts of basic and nonbasic variables.

**Definition**  
After the Gauss–Jordan method has been applied to any linear system, a variable that appears with a coefficient of 1 in a single equation and a coefficient of 0 in all other equations is called a **basic variable** (BV).

Any variable that is not a basic variable is called a **nonbasic variable** (NBV).

Let BV be the set of basic variables for \( A'x = b' \) and NBV be the set of nonbasic variables for \( A'x = b' \). The character of the solutions to \( A'x = b' \) depends on which of the following cases occurs.

**Case 1**  
\( A'x = b' \) has at least one row of form \([0 \ 0 \ \cdots \ 0 \ c]\) \((c \neq 0)\). Then \( Ax = b \) has no solution (recall Example 6). As an example of Case 1, suppose that when the Gauss–Jordan method is applied to the system \( Ax = b \), the following matrix is obtained:
\[ A' \mathbf{b'} = \begin{bmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 3 \\
0 & 0 & 0 & 2
\end{bmatrix} \]

In this case, \( A' \mathbf{x} = \mathbf{b'} \) (and \( Ax = b \)) has no solution.

**Case 2** Suppose that Case 1 does not apply and NBV, the set of nonbasic variables, is empty. Then \( A' \mathbf{x} = \mathbf{b'} \) (and \( Ax = b \)) will have a unique solution. To illustrate this, we recall that in solving

\[
\begin{align*}
2x_1 + 2x_2 + x_3 &= 9 \\
2x_1 - x_2 + 2x_3 &= 6 \\
x_1 - x_2 + 2x_3 &= 5
\end{align*}
\]

the Gauss–Jordan method yielded

\[ A' \mathbf{b'} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \]

In this case, BV = \( \{x_1, x_2, x_3\} \) and NBV is empty. Then the unique solution to \( A' \mathbf{x} = \mathbf{b'} \) (and \( Ax = b \)) is \( x_1 = 1, x_2 = 2, x_3 = 3 \).

**Case 3** Suppose that Case 1 does not apply and NBV is nonempty. Then \( A' \mathbf{x} = \mathbf{b'} \) (and \( Ax = b \)) will have an infinite number of solutions. To obtain these, first assign each nonbasic variable an arbitrary value. Then solve for the value of each basic variable in terms of the nonbasic variables. For example, suppose

\[ A' \mathbf{b'} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \]  \hspace{1cm} (15)

Because Case 1 does not apply, and BV = \( \{x_1, x_2, x_3\} \) and NBV = \( \{x_4, x_5\} \), we have an example of Case 3: \( A' \mathbf{x} = \mathbf{b'} \) (and \( Ax = b \)) will have an infinite number of solutions. To see what these solutions look like, write down \( A' \mathbf{x} = \mathbf{b'} \):

\[
\begin{align*}
x_1 + x_4 + x_5 &= 3 \hspace{1cm} (15.1) \\
x_2 + 2x_4 &= 2 \hspace{1cm} (15.2) \\
x_3 + x_5 &= 1 \hspace{1cm} (15.3) \\
0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 &= 0 \hspace{1cm} (15.4)
\end{align*}
\]

Now assign the nonbasic variables \( (x_4 \text{ and } x_5) \) arbitrary values \( c \) and \( k \), with \( x_4 = c \) and \( x_5 = k \). From (15.1), we find that \( x_1 = 3 - c - k \). From (15.2), we find that \( x_2 = 2 - 2c \). From (15.3), we find that \( x_3 = 1 - k \). Because (15.4) holds for all values of the variables, \( x_1 = 3 - c - k, x_2 = 2 - 2c, x_3 = 1 - k, x_4 = c, \) and \( x_5 = k \) will, for any values of \( c \) and \( k \), be a solution to \( A' \mathbf{x} = \mathbf{b'} \) (and \( Ax = b \)).

Our discussion of the Gauss–Jordan method is summarized in Figure 6. We have devoted so much time to the Gauss–Jordan method because, in our study of linear programming, examples of Case 3 (linear systems with an infinite number of solutions) will occur repeatedly. Because the end result of the Gauss–Jordan method must always be one of Cases 1–3, we have shown that any linear system will have no solution, a unique solution, or an infinite number of solutions.

**2.3 The Gauss–Jordan Method for Solving Systems of Linear Equations**
Figure 6
Description of Gauss-Jordan Method for Solving Linear Equations

Problems

Group A

Use the Gauss–Jordan method to determine whether each of the following linear systems has no solution, a unique solution, or an infinite number of solutions. Indicate the solutions (if any exist).

1 \[ x_1 + x_2 + x_4 = 3 \]
   \[ x_3 = 4 \]
   \[ x_1 + 2x_2 + x_3 + x_4 = 8 \]

2 \[ x_1 + x_2 + x_3 = 4 \]
   \[ x_1 + 2x_2 = 6 \]

3 \[ x_1 + x_2 = 1 \]
   \[ 2x_1 + x_2 = 3 \]
   \[ 3x_1 + 2x_2 = 4 \]

4 \[ 2x_1 - x_2 + x_3 + x_4 = 6 \]
   \[ x_1 + x_2 + x_3 + x_4 = 4 \]

5 \[ x_1 + x_4 = 5 \]
   \[ x_2 + 2x_4 = 5 \]
   \[ x_3 + 0.5x_4 = 1 \]
   \[ 2x_3 + x_4 = 3 \]

6 \[ 2x_2 + 2x_3 = 4 \]
   \[ x_1 + 2x_3 + x_3 = 4 \]
   \[ x_2 - x_3 = 0 \]

7 \[ x_1 + x_2 = 2 \]
   \[ -x_2 + 2x_3 = 3 \]
   \[ x_2 + x_3 = 3 \]

8 \[ x_1 + x_2 + x_3 = 1 \]
   \[ x_2 + 2x_3 + x_4 = 2 \]
   \[ x_4 = 3 \]

Group B

9 Suppose that a linear system \( Ax = b \) has more variables than equations. Show that \( Ax = b \) cannot have a unique solution.

2.4 Linear Independence and Linear Dependence†

In this section, we discuss the concepts of a linearly independent set of vectors, a linearly dependent set of vectors, and the rank of a matrix. These concepts will be useful in our study of matrix inverses.

Before defining a linearly independent set of vectors, we need to define a linear combination of a set of vectors. Let \( V = \{ v_1, v_2, \ldots, v_n \} \) be a set of row vectors all of which have the same dimension.

†This section covers topics that may be omitted with no loss of continuity.
**Definition** A linear combination of the vectors in $V$ is any vector of the form $c_1v_1 + c_2v_2 + \cdots + c_kv_k$, where $c_1, c_2, \ldots, c_k$ are arbitrary scalars.

For example, if $V = \{(1, 2), (2, 1)\}$, then

\[
\begin{align*}
2v_1 - v_2 &= 2(1, 2) - (2, 1) = (0, 3) \\
v_1 + 3v_2 &= (1, 2) + 3(2, 1) = (7, 5) \\
0v_1 + 3v_2 &= (0, 0) + 3(2, 1) = (6, 3)
\end{align*}
\]

are linear combinations of vectors in $V$. The foregoing definition may also be applied to a set of column vectors.

Suppose we are given a set $V = \{v_1, v_2, \ldots, v_k\}$ of $m$-dimensional row vectors. Let $0 = [0 \ 0 \ \cdots \ 0]$ be the $m$-dimensional 0 vector. To determine whether $V$ is a linearly independent set of vectors, we try to find a linear combination of the vectors in $V$ that adds up to $0$. Clearly, $0v_1 + 0v_2 + \cdots + 0v_k$ is a linear combination of vectors in $V$ that adds up to $0$. We call the linear combination of vectors in $V$ for which $c_1 = c_2 = \cdots = c_k = 0$ the trivial linear combination of vectors in $V$. We may now define linearly independent and linearly dependent sets of vectors.

**Definition** A set $V$ of $m$-dimensional vectors is linearly independent if the only linear combination of vectors in $V$ that equals $0$ is the trivial linear combination.

A set $V$ of $m$-dimensional vectors is linearly dependent if there is a nontrivial linear combination of the vectors in $V$ that adds up to $0$.

The following examples should clarify these definitions.

**Example 8** 0 Vector Makes Set LD

Show that any set of vectors containing the 0 vector is a linearly dependent set.

**Solution** To illustrate, we show that if $V = \{(0, 0), (1, 0), (0, 1)\}$, then $V$ is linearly dependent, because if, say, $c_1 \neq 0$, then $c_1(0, 0) + 0(1, 0) + 0(0, 1) = (0, 0)$. Thus, there is a nontrivial linear combination of vectors in $V$ that adds up to 0.

**Example 9** LI Set of Vectors

Show that the set of vectors $V = \{(1, 0), (0, 1)\}$ is a linearly independent set of vectors.

**Solution** We try to find a nontrivial linear combination of the vectors in $V$ that yields $0$. This requires that we find scalars $c_1$ and $c_2$ (at least one of which is nonzero) satisfying $c_1(1, 0) + c_2(0, 1) = (0, 0)$. Thus, $c_1$ and $c_2$ must satisfy $c_1 = c_2 = 0$. This implies $c_1 = c_2 = 0$. The only linear combination of vectors in $V$ that yields $0$ is the trivial linear combination. Therefore, $V$ is a linearly independent set of vectors.

**Example 10** LD Set of Vectors

Show that $V = \{(1, 2), (2, 4)\}$ is a linearly dependent set of vectors.

**Solution** Because $2(1, 2) - 1(2, 4) = (0, 0)$, there is a nontrivial linear combination with $c_1 = 2$ and $c_2 = -1$ that yields $0$. Thus, $V$ is a linearly dependent set of vectors.

Intuitively, what does it mean for a set of vectors to be linearly dependent? To understand the concept of linear dependence, observe that a set of vectors $V$ is linearly dependent (as
long as 0 is not in \( V \) if and only if some vector in \( V \) can be written as a nontrivial linear combination of other vectors in \( V \) (see Problem 9 at the end of this section). For instance, in Example 10, \( [2, 4] = 2[1, 2] \). Thus, if a set of vectors \( V \) is linearly dependent, the vectors in \( V \) are, in some way, not all "different" vectors. By "different" we mean that the direction specified by any vector in \( V \) cannot be expressed by adding together multiples of other vectors in \( V \). For example, in two dimensions it can be shown that two vectors are linearly dependent if and only if they lie on the same line (see Figure 7).

**The Rank of a Matrix**

The Gauss–Jordan method can be used to determine whether a set of vectors is linearly independent or linearly dependent. Before describing how this is done, we define the concept of the rank of a matrix.

Let \( A \) be any \( m \times n \) matrix, and denote the rows of \( A \) by \( r_1, r_2, \ldots, r_m \). Also define \( R = \{r_1, r_2, \ldots, r_m\} \).

**Definition**

The rank of \( A \) is the number of vectors in the largest linearly independent subset of \( R \).

The following three examples illustrate the concept of rank.

**Example 1.1 Matrix with 0 Rank**

Show that rank \( A = 0 \) for the following matrix:

\[
A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\]

**Solution** For the set of vectors \( R = \{[0, 0], [0, 0]\} \), it is impossible to choose a subset of \( R \) that is linearly independent (recall Example 8).

**Example 1.2 Matrix with Rank of 1**

Show that rank \( A = 1 \) for the following matrix:

\[
A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}
\]

34
Solution Here \( R = \{[1 \quad 1], [2 \quad 2]\} \). The set \( \{[1 \quad 1]\} \) is a linearly independent subset of \( R \), so rank \( A \) must be at least 1. If we try to find two linearly independent vectors in \( R \), we fail because 
\[
2([1 \quad 1]) - [2 \quad 2] = [0 \quad 0].
\]
This means that rank \( A \) cannot be 2. Thus, rank \( A \) must equal 1.

### Example 13
**Matrix with Rank of 2**

Show that rank \( A = 2 \) for the following matrix:

\[
A = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

**Solution** Here \( R = \{[1 \quad 0], [0 \quad 1]\} \). From Example 9, we know that \( R \) is a linearly independent set of vectors. Thus, rank \( A = 2 \).

---

To find the rank of a given matrix \( A \), simply apply the Gauss–Jordan method to the matrix \( A \). Let the final result be the matrix \( \overline{A} \). It can be shown that performing a sequence of EROs on a matrix does not change the rank of the matrix. This implies that rank \( A = \) rank \( \overline{A} \). It is also apparent that the rank of \( A \) will be the number of nonzero rows in \( A \). Combining these facts, we find that rank \( A = \) rank \( \overline{A} = \) number of nonzero rows in \( A \).

### Example 14
**Using Gauss–Jordan Method to Find Rank of Matrix**

Find

\[
\text{rank } A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 3 \end{bmatrix}
\]

**Solution** The Gauss–Jordan method yields the following sequence of matrices:

\[
A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 \frac{1}{2} & \frac{1}{2} \\ 0 & 2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}
\]

Thus, rank \( A = \) rank \( \overline{A} = 3 \).

---

**How to Tell Whether a Set of Vectors Is Linearly Independent**

We now describe a method for determining whether a set of vectors \( V = \{v_1, v_2, \ldots, v_m\} \) is linearly independent.

Form the matrix \( A \) whose \( i \)th row is \( v_i \). \( A \) will have \( m \) rows. If rank \( A = m \), then \( V \) is a linearly independent set of vectors, whereas if rank \( A < m \), then \( V \) is a linearly dependent set of vectors.

### Example 15
**A Linearly Dependent Set of Vectors**

Determine whether \( V = \{[1 \quad 0 \quad 0], [0 \quad 1 \quad 0], [1 \quad 1 \quad 0]\} \) is a linearly independent set of vectors.
Solution  The Gauss–Jordan method yields the following sequence of matrices:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
1 & 1 & 0
\end{bmatrix} \rightarrow
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0
\end{bmatrix} = \tilde{A}
\]

Thus, rank \( A = \text{rank} \tilde{A} = 2 < 3 \). This shows that \( V \) is a linearly dependent set of vectors. In fact, the EROs used to transform \( A \) to \( \tilde{A} \) can be used to show that \([1 \ 1 \ 0] = [1 \ 0 \ 0] + [0 \ 1 \ 0] \). This equation also shows that \( V \) is a linearly dependent set of vectors.

---

PROBLEMS

Group A

Determine if each of the following sets of vectors is linearly independent or linearly dependent.

1 \( V = \{(1 \ 0 \ 1), (1 \ 2 \ 1), (2 \ 2 \ 2)\} \)

2 \( V = \{(2 \ 1 \ 0), (1 \ 2 \ 0), (3 \ 3 \ 1)\} \)

3 \( V = \{(2 \ 1), (1 \ 2)\} \)

4 \( V = \{(2 \ 0), (3 \ 0)\} \)

5 \( V = \begin{bmatrix}
1 & 4 & 5 \\
2 & 5 & 7 \\
3 & 6 & 9
\end{bmatrix} \)

6 \( V = \begin{bmatrix}
1 & 0 & 1 \\
0 & 2 & 0 \\
0 & 1 & 1
\end{bmatrix} \)

---

Group B

7 Show that the linear system \( Ax = b \) has a solution if and only if \( b \) can be written as a linear combination of the columns of \( A \).

8 Suppose there is a collection of three or more two-dimensional vectors. Provide an argument showing that the collection must be linearly dependent.

9 Show that a set of vectors \( V \) (not containing the 0 vector) is linearly dependent if and only if there exists some vector in \( V \) that can be written as a nontrivial linear combination of other vectors in \( V \).

---

2.5 The Inverse of a Matrix

To solve a single linear equation such as \( 4x = 3 \), we simply multiply both sides of the equation by the multiplicative inverse of 4, which is \( 4^{-1} \), or \( \frac{1}{4} \). This yields \( 4^{-1}(4x) = (4^{-1})3 \), or \( x = \frac{3}{4} \). (Of course, this method fails to work for the equation \( 0x = 3 \), because zero has no multiplicative inverse.) In this section, we develop a generalization of this technique that can be used to solve “square” (number of equations = number of unknowns) linear systems. We begin with some preliminary definitions.

**Definition.** A square matrix is any matrix that has an equal number of rows and columns.

The diagonal elements of a square matrix are those elements \( a_{ii} \) such that \( i = j \).

A square matrix for which all diagonal elements are equal to 1 and all nondiagonal elements are equal to 0 is called an identity matrix.

The \( m \times m \) identity matrix will be written as \( I_m \). Thus,

\[
I_2 = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}, \quad I_3 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad \ldots
\]
If the multiplications $I_m A$ and $A I_m$ are defined, it is easy to show that $I_m A = A I_m = A$. Thus, just as the number 1 serves as the unit element for multiplication of real numbers, $I_m$ serves as the unit element for multiplication of matrices.

Recall that $\frac{1}{2}$ is the multiplicative inverse of 2. This is because $4 \cdot \left( \frac{1}{2} \right) = \left( \frac{1}{2} \right) 4 = 1$. This motivates the following definition of the inverse of a matrix.

**Definition**

For a given $m \times m$ matrix $A$, the $m \times m$ matrix $B$ is the **inverse** of $A$ if

$$BA = AB = I_m$$

(16)

(It can be shown that if $BA = I_m$ or $AB = I_m$, then the other quantity will also equal $I_m$.)

Some square matrices do not have inverses. If there does exist an $m \times m$ matrix $B$ that satisfies Equation (16), then we write $B = A^{-1}$. For example, if

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

the reader can verify that

$$\begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -5 & 1 & -7 \\ 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 & 1 \\ -5 & 1 & -7 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus,

$$A^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ -5 & 1 & -7 \\ 1 & 0 & 2 \end{bmatrix}$$

To see why we are interested in the concept of a matrix inverse, suppose we want to solve a linear system $Ax = b$ that has $m$ equations and $m$ unknowns. Suppose that $A^{-1}$ exists. Multiplying both sides of $Ax = b$ by $A^{-1}$, we see that any solution of $Ax = b$ must also satisfy $A^{-1} (Ax) = A^{-1} b$. Using the associative law and the definition of a matrix inverse, we obtain

$$(A^{-1} A) x = A^{-1} b$$

or

$$I_m x = A^{-1} b$$

or

$$x = A^{-1} b$$

This shows that knowing $A^{-1}$ enables us to find the unique solution to a square linear system. This is the analog of solving $4x = 3$ by multiplying both sides of the equation by $4^{-1}$.

The Gauss–Jordan method may be used to find $A^{-1}$ (or to show that $A^{-1}$ does not exist). To illustrate how we can use the Gauss–Jordan method to invert a matrix, suppose we want to find $A^{-1}$ for

$$A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

2.5 The Inverse of a Matrix
This requires that we find a matrix

\[
\begin{bmatrix}
    a & b \\
    c & d
\end{bmatrix} = A^{-1}
\]

that satisfies

\[
\begin{bmatrix}
    2 & 5 \\
    1 & 3
\end{bmatrix}
\begin{bmatrix}
    a & b \\
    c & d
\end{bmatrix} =
\begin{bmatrix}
    1 & 0 \\
    0 & 1
\end{bmatrix}
\] \tag{17}

From Equation (17), we obtain the following pair of simultaneous equations that must be satisfied by $a$, $b$, $c$, and $d$:

\[
\begin{bmatrix}
    2 & 5 \\
    1 & 3
\end{bmatrix}
\begin{bmatrix}
    a \\
    c
\end{bmatrix} =
\begin{bmatrix}
    1 \\
    0
\end{bmatrix} \quad \begin{bmatrix}
    2 & 5 \\
    1 & 3
\end{bmatrix}
\begin{bmatrix}
    b \\
    d
\end{bmatrix} =
\begin{bmatrix}
    0 \\
    1
\end{bmatrix}
\]

Thus, to find

\[
\begin{bmatrix}
    a \\
    c
\end{bmatrix}
\]

(the first column of $A^{-1}$), we can apply the Gauss–Jordan method to the augmented matrix

\[
\begin{bmatrix}
    2 & 5 & 1 \\
    1 & 3 & 0
\end{bmatrix}
\]

Once EROs have transformed

\[
\begin{bmatrix}
    2 & 5 \\
    1 & 3
\end{bmatrix}
\]

to $I_2$,

\[
\begin{bmatrix}
    1 \\
    0
\end{bmatrix}
\]

will have been transformed into the first column of $A^{-1}$. To determine

\[
\begin{bmatrix}
    b \\
    d
\end{bmatrix}
\]

(the second column of $A^{-1}$), we apply EROs to the augmented matrix

\[
\begin{bmatrix}
    2 & 5 & 0 \\
    1 & 3 & 1
\end{bmatrix}
\]

When

\[
\begin{bmatrix}
    2 & 5 \\
    1 & 3
\end{bmatrix}
\]

has been transformed into $I_2$,

\[
\begin{bmatrix}
    0 \\
    1
\end{bmatrix}
\]

will have been transformed into the second column of $A^{-1}$. Thus, to find each column of $A^{-1}$, we must perform a sequence of EROs that transform

\[
\begin{bmatrix}
    2 & 5 \\
    1 & 3
\end{bmatrix}
\]
into $I_2$. This suggests that we can find $A^{-1}$ by applying EROs to the $2 \times 4$ matrix

$$A|I_2 = \begin{bmatrix} 2 & 5 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{bmatrix}$$

When

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

has been transformed to $I_2$,

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

will have been transformed into the first column of $A^{-1}$, and

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

will have been transformed into the second column of $A^{-1}$. Thus, as $A$ is transformed into $I_2$, $I_2$ is transformed into $A^{-1}$. The computations to determine $A^{-1}$ follow.

**Step 1**  Multiply row 1 of $A|I_2$ by $\frac{1}{2}$. This yields

$$A'\left| I'_2 \right. = \begin{bmatrix} 1 & \frac{5}{2} & \frac{1}{2} & 0 \\ 1 & 3 & 0 & 1 \end{bmatrix}$$

**Step 2**  Replace row 2 of $A'|I'_2$ by $-1$(row 1 of $A'|I'_2$) + row 2 of $A'|I'_2$. This yields

$$A''|I''_2 = \begin{bmatrix} 1 & \frac{5}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}$$

**Step 3**  Multiply row 2 of $A''|I''_2$ by 2. This yields

$$A'''|I'''_2 = \begin{bmatrix} 1 & \frac{5}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -1 & 2 \end{bmatrix}$$

**Step 4**  Replace row 1 of $A'''|I'''_2$ by $\frac{5}{2}$(row 2 of $A'''|I'''_2$) + row 1 of $A'''|I'''_2$. This yields

$$\begin{bmatrix} 1 & 0 & 3 & -5 \\ 0 & 1 & -1 & 2 \end{bmatrix}$$

Because $A$ has been transformed into $I_2$, $I_2$ will have been transformed into $A^{-1}$. Hence,

$$A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

The reader should verify that $AA^{-1} = A^{-1}A = I_2$.

**A Matrix May Not Have an Inverse**

Some matrices do not have inverses. To illustrate, let

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

(18)
To find $A^{-1}$ we must solve the following pair of simultaneous equations:

\[
\begin{bmatrix}
1 & 2 \\
2 & 4
\end{bmatrix}
\begin{bmatrix}
e \\
g
\end{bmatrix}
=
\begin{bmatrix}
1 \\
0
\end{bmatrix}
\]  \hfill (18.1)

\[
\begin{bmatrix}
1 & 2 \\
2 & 4
\end{bmatrix}
\begin{bmatrix}
f \\
h
\end{bmatrix}
=
\begin{bmatrix}
0 \\
1
\end{bmatrix}
\]  \hfill (18.2)

When we try to solve (18.1) by the Gauss-Jordan method, we find that

\[
\begin{bmatrix}
1 & 2 & | & 1 \\
2 & 4 & | & 0
\end{bmatrix}
\]

is transformed into

\[
\begin{bmatrix}
1 & 2 & | & 1 \\
0 & 0 & | & -2
\end{bmatrix}
\]

This indicates that (18.1) has no solution, and $A^{-1}$ cannot exist.

Observe that (18.1) fails to have a solution, because the Gauss-Jordan method transforms $A$ into a matrix with a row of zeros on the bottom. This can only happen if rank $A < 2$. If $m \times m$ matrix $A$ has rank $A < m$, then $A^{-1}$ will not exist.

### The Gauss-Jordan Method for Inverting an $m \times m$ Matrix $A$

**Step 1** Write down the $m \times 2m$ matrix $A|I_m$.

**Step 1** Use EROs to transform $A|I_m$ into $I_m|B$. This will be possible only if rank $A = m$. In this case, $B = A^{-1}$. If rank $A < m$, then $A$ has no inverse.

### Using Matrix Inverses to Solve Linear Systems

As previously stated, matrix inverses can be used to solve a linear system $Ax = b$ in which the number of variables and equations are equal. Simply multiply both sides of $Ax = b$ by $A^{-1}$ to obtain the solution $x = A^{-1}b$. For example, to solve

\[
2x_1 + 5x_2 = 7 \\
x_1 + 3x_2 = 4
\]

write the matrix representation of (19):

\[
\begin{bmatrix}
2 & 5 \\
1 & 3
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
=
\begin{bmatrix}
7 \\
4
\end{bmatrix}
\]  \hfill (20)

Let

\[
A = \begin{bmatrix}
2 & 5 \\
1 & 3
\end{bmatrix}
\]

We found in the previous illustration that

\[
A^{-1} = \begin{bmatrix}
3 & -5 \\
-1 & 2
\end{bmatrix}
\]
Multiplying both sides of (20) by $A^{-1}$, we obtain

$$
\begin{bmatrix}
3 & -5 \\
-1 & 2
\end{bmatrix}
\begin{bmatrix}
2 & 5 \\
3 & 1
\end{bmatrix}
= 
\begin{bmatrix}
3 & -5 \\
-1 & 2
\end{bmatrix}
\begin{bmatrix}
1 \\
1
\end{bmatrix}
$$

Thus, $x_1 = 1$, $x_2 = 1$ is the unique solution to system (19).

**Inverting Matrices with EXCEL**

The EXCEL = MINVERSE command makes it easy to invert a matrix. See Figure 8 and file Minverse.xls. Suppose we want to invert the matrix

$$
A = \begin{bmatrix}
2 & 0 & -1 \\
3 & 1 & 2 \\
-1 & 0 & 1
\end{bmatrix}
$$

Simply enter the matrix in E3:G5 and select the range (we chose E7:G9) where you want $A^{-1}$ to be computed. In the upper left-hand corner of the range E7:G9 (cell E7) we enter the formula

$$
= \text{MINVERSE(E3:G5)}
$$

and select Control Shift Enter. This enters an “array function” that computes $A^{-1}$ in the range E7:G9. You cannot edit part of an array function, so if you want to delete $A^{-1}$, you must delete the entire range where $A^{-1}$ is present.

**PROBLEMS**

**Group A**

Find $A^{-1}$ (if it exists) for the following matrices:

1. $\begin{bmatrix}
1 & 3 \\
2 & 5
\end{bmatrix}$
2. $\begin{bmatrix}
4 & 1 & -2 \\
3 & 1 & -1
\end{bmatrix}$
3. $\begin{bmatrix}
1 & 0 & 1 \\
1 & 1 & 1 \\
2 & 1 & 2
\end{bmatrix}$

5. Use the answer to Problem 1 to solve the following linear system:

$$
\begin{align*}
x_1 + 3x_2 &= 4 \\
2x_1 + 5x_2 &= 7
\end{align*}
$$

6. Use the answer to Problem 2 to solve the following linear system:

$$
\begin{align*}
x_1 + x_3 &= 4 \\
4x_1 + x_2 - 2x_3 &= 0 \\
3x_1 + x_2 - x_3 &= 2
\end{align*}
$$

**Group B**

7. Show that a square matrix has an inverse if and only if its rows form a linearly independent set of vectors.

8. Consider a square matrix $B$ whose inverse is given by $B^{-1}$. In terms of $B^{-1}$, what is the inverse of the matrix $100B$?
b Let \( B' \) be the matrix obtained from \( B \) by doubling every entry in row 1 of \( B \). Explain how we could obtain the inverse of \( B' \) from \( B^{-1} \).

c Let \( B' \) be the matrix obtained from \( B \) by doubling every entry in column 1 of \( B \). Explain how we could obtain the inverse of \( B' \) from \( B^{-1} \).

9 Suppose that \( A \) and \( B \) both have inverses. Find the inverse of the matrix \( AB \).

10 Suppose \( A \) has an inverse. Show that \((A^T)^{-1} = (A^{-1})^T\).
(Hint: Use the fact that \( AA^{-1} = I \), and take the transpose of both sides.)

11 A square matrix \( A \) is **orthogonal** if \( AA^T = I \). What properties must be possessed by the columns of an orthogonal matrix?

### 2.6 Determinants

Associated with any square matrix \( A \) is a number called the **determinant** of \( A \) (often abbreviated as \( \text{det} \ A \) or \( |A| \)). Knowing how to compute the determinant of a square matrix will be useful in our study of nonlinear programming.

For a \( 1 \times 1 \) matrix \( A = [a_{11}] \),

\[
\text{det} \ A = a_{11}
\]  \hspace{1cm} (21)

For a \( 2 \times 2 \) matrix

\[
A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}
\]

\[
\text{det} \ A = a_{11}a_{22} - a_{12}a_{21}
\]  \hspace{1cm} (22)

For example,

\[
\text{det} \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix} = 2(5) - 3(4) = -2
\]

Before we learn how to compute \( \text{det} \ A \) for larger square matrices, we need to define the concept of the **minor** of a matrix.

**Definition** If \( A \) is an \( m \times m \) matrix, then for any values of \( i \) and \( j \), the \( ij \)th minor of \( A \) (written \( A_{ij} \)) is the \((m - 1) \times (m - 1)\) submatrix of \( A \) obtained by deleting row \( i \) and column \( j \) of \( A \).

For example,

If \( A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \), then \( A_{12} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} \) and \( A_{32} = \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix} \).

Let \( A \) be any \( m \times m \) matrix. We may write \( A \) as

\[
A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix}
\]

To compute \( \text{det} \ A \), pick any value of \( i \) (\( i = 1, 2, \ldots, m \)) and compute \( \text{det} \ A \):

\[
\text{det} \ A = (-1)^{i+1}a_{i1}(\text{det} \ A_{i1}) + (-1)^{i+2}a_{i2}(\text{det} \ A_{i2}) + \cdots + (-1)^{i+m}a_{im}(\text{det} \ A_{im})
\]  \hspace{1cm} (23)
Formula (23) is called the expansion of det $A$ by the cofactors of row $i$. The virtue of (23) is that it reduces the computation of det $A$ for an $m \times m$ matrix to computations involving only $(m - 1) \times (m - 1)$ matrices. Apply (23) until det $A$ can be expressed in terms of $2 \times 2$ matrices. Then use Equation (22) to find the determinants of the relevant $2 \times 2$ matrices.

To illustrate the use of (23), we find det $A$ for

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

We expand det $A$ by using row 1 cofactors. Notice that $a_{11} = 1$, $a_{12} = 2$, and $a_{13} = 3$. Also

$$A_{11} = \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}$$

so by (22), det $A_{11} = 5(9) - 8(6) = -3$;

$$A_{12} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}$$

so by (22), det $A_{12} = 4(9) - 7(6) = -6$; and

$$A_{13} = \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}$$

so by (22), det $A_{13} = 4(8) - 7(5) = -3$. Then by (23),

$$\det A = (-1)^{1+1}a_{11}(\det A_{11}) + (-1)^{1+2}a_{12}(\det A_{12}) + (-1)^{1+3}a_{13}(\det A_{13})$$

$$= (1)(1)(-3) + (-1)(2)(-6) + (1)(3)(-3) = -3 + 12 - 9 = 0$$

The interested reader may verify that expansion of det $A$ by either row 2 or row 3 cofactors also yields det $A = 0$.

We close our discussion of determinants by noting that they can be used to invert square matrices and to solve linear equation systems. Because we already have learned to use the Gauss–Jordan method to invert matrices and to solve linear equation systems, we will not discuss these uses of determinants.

**Problems**

**Group A**

1. Verify that det
   $$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = 0$$
   by using expansions by row 2 and row 3 cofactors.

2. Find det
   $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

3. A matrix is said to be upper triangular if for $i > j$, $a_{ij} = 0$. Show that the determinant of any upper triangular $3 \times 3$ matrix is equal to the product of the matrix's diagonal elements. (This result is true for any upper triangular matrix.)

**Group B**

4. a. Show that for any $1 \times 1$ and $3 \times 3$ matrix, det $-A = -\det A$.

   b. Show that for any $2 \times 2$ and $4 \times 4$ matrix, det $-A = \det A$.

   c. Generalize the results of parts (a) and (b).
Matrices

A matrix is any rectangular array of numbers. For the matrix $A$, we let $a_{ij}$ represent the element of $A$ in row $i$ and column $j$.

A matrix with only one row or one column may be thought of as a vector. Vectors appear in boldface type ($\mathbf{v}$). Given a row vector $\mathbf{u} = [u_1 \ u_2 \ \cdots \ u_n]$ and a column

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

of the same dimension, the scalar product of $\mathbf{u}$ and $\mathbf{v}$ (written $\mathbf{u} \cdot \mathbf{v}$) is the number $u_1v_1 + u_2v_2 + \cdots + u_nv_n$.

Given two matrices $A$ and $B$, the matrix product of $A$ and $B$ (written $AB$) is defined if and only if the number of columns in $A =$ the number of rows in $B$. Suppose this is the case and $A$ has $m$ rows and $B$ has $n$ columns. Then the matrix product $C = AB$ of $A$ and $B$ is the $m \times n$ matrix $C$ whose $ij$th element is determined as follows: The $ij$th element of $C =$ the scalar product of row $i$ of $A$ with column $j$ of $B$.

Matrices and Linear Equations

The linear equation system

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots \quad \vdots \quad \vdots \quad = \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

may be written as $Ax = b$ or $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

The Gauss–Jordan Method

Using elementary row operations (EROs), we may solve any linear equation system. From a matrix $A$, an ERO yields a new matrix $A'$ via one of three procedures.

Type 1 ERO

Obtain $A'$ by multiplying any row of $A$ by a nonzero scalar.

Type 2 ERO

Multiply any row of $A$ (say, row $i$) by a nonzero scalar $c$. For some $j \neq i$, let row $j$ of $A' = c$(row $i$ of $A$) + row $j$ of $A$, and let the other rows of $A'$ be the same as the rows of $A$.  44
Type 3 ERO
Interchange any two rows of $A$.

The Gauss–Jordan method uses EROs to solve linear equation systems, as shown in the following steps.

**Step 1** To solve $Ax = b$, write down the augmented matrix $A|b$.

**Step 2** Begin with row 1 as the current row, column 1 as the current column, and $a_{11}$ as the current entry. (a) If $a_{11}$ (the current entry) is nonzero, then use EROs to transform column 1 (the current column) to

$$
\begin{bmatrix}
1 \\
0 \\
\vdots \\
0
\end{bmatrix}
$$

Then obtain the new current row, column, and entry by moving down one row and one column to the right, and go to step 3. (b) If $a_{11}$ (the current entry) equals 0, then do a Type 3 ERO switch with any row with a nonzero value in the same column. Use EROs to transform column 1 to

$$
\begin{bmatrix}
1 \\
0 \\
\vdots \\
0
\end{bmatrix}
$$

and proceed to step 3 after moving into a new current row, column, and entry. (c) If there are no nonzero numbers in the first column, then proceed to a new current column and entry. Then go to step 3.

**Step 3** (a) If the current entry is nonzero, use EROs to transform it to 1 and the rest of the current column’s entries to 0. Obtain the new current row, column, and entry. If this is impossible, then stop. Otherwise, repeat step 3. (b) If the current entry is 0, then do a Type 3 ERO switch with any row with a nonzero value in the same column. Transform the column using EROs and move to the next current entry. If this is impossible, then stop. Otherwise, repeat step 3. (c) If the current column has no nonzero numbers below the current row, then obtain the new current column and entry, and repeat step 3. If it is impossible, then stop.

This procedure may require “passing over” one or more columns without transforming them.

**Step 4** Write down the system of equations $A'x = b'$ that corresponds to the matrix $A'|b'$ obtained when step 3 is completed. Then $A'x = b'$ will have the same set of solutions as $Ax = b$.

To describe the set of solutions to $A'x = b'$ (and $Ax = b$), we define the concepts of basic and nonbasic variables. After the Gauss–Jordan method has been applied to any linear system, a variable that appears with a coefficient of 1 in a single equation and a coefficient of 0 in all other equations is called a **basic variable**. Any variable that is not a basic variable is called a **nonbasic variable**.
Let $B \mathbf{V}$ be the set of basic variables for $A' \mathbf{x} = \mathbf{b'}$ and NBV be the set of nonbasic variables for $A' \mathbf{x} = \mathbf{b'}$.

**Case 1**  
$A' \mathbf{x} = \mathbf{b'}$ contains at least one row of the form $[0 \quad 0 \quad \cdots \quad 0 | c](c \neq 0)$. In this case, $A \mathbf{x} = \mathbf{b}$ has no solution.

**Case 2**  
If Case 1 does not apply and NBV, the set of nonbasic variables, is empty, then $A \mathbf{x} = \mathbf{b}$ will have a unique solution.

**Case 3**  
If Case 1 does not hold and NBV is nonempty, then $A \mathbf{x} = \mathbf{b}$ will have an infinite number of solutions.

### Linear Independence, Linear Dependence, and the Rank of a Matrix

A set $V$ of $m$-dimensional vectors is **linearly independent** if the only linear combination of vectors in $V$ that equals $\mathbf{0}$ is the trivial linear combination. A set $V$ of $m$-dimensional vectors is **linearly dependent** if there is a nontrivial linear combination of the vectors in $V$ that adds to $\mathbf{0}$.

Let $A$ be any $m \times n$ matrix, and denote the rows of $A$ by $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$. Also define $R = \{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m\}$. The **rank** of $A$ is the number of vectors in the largest linearly independent subset of $R$. To find the rank of a given matrix $A$, apply the Gauss–Jordan method to the matrix $A$. Let the final result be the matrix $\tilde{A}$. Then $\text{rank } A = \text{rank } \tilde{A} =$ number of nonzero rows in $\tilde{A}$.

To determine if a set of vectors $V = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly dependent, form the matrix $A$ whose $i$th row is $\mathbf{v}_i$. $A$ will have $m$ rows. If $\text{rank } A = m$, then $V$ is a linearly independent set of vectors; if $\text{rank } A < m$, then $V$ is a linearly dependent set of vectors.

### Inverse of a Matrix

For a given square $(m \times m)$ matrix $A$, if $AB = BA = I_m$ then $B$ is the **inverse** of $A$ (written $B = A^{-1}$). The Gauss–Jordan method for inverting an $m \times m$ matrix $A$ to get $A^{-1}$ is as follows:

**Step 1**  
Write down the $m \times 2m$ matrix $A|I_m$.

**Step 2**  
Use EROs to transform $A|I_m$ into $I_m|B$. This will only be possible if rank $A = m$. In this case, $B = A^{-1}$. If rank $A < m$, then $A$ has no inverse.

### Determinants

Associated with any square $(m \times m)$ matrix $A$ is a number called the **determinant** of $A$ (written $\det A$ or $|A|$). For a $1 \times 1$ matrix, $\det A = a_{11}$. For a $2 \times 2$ matrix, $\det A = a_{11}a_{22} - a_{12}a_{21}$. For a general $m \times m$ matrix, we can find $\det A$ by repeated application of the following formula (valid for $i = 1, 2, \ldots, m$):

$$
\det A = (-1)^{i+1}a_{i1}(\det A_{i1}) + (-1)^{i+2}a_{i2}(\det A_{i2}) + \cdots + (-1)^{i+m}a_{im}(\det A_{im})
$$

Here $A_{ij}$ is the $i$th minor of $A$, which is the $(m - 1) \times (m - 1)$ matrix obtained from $A$ after deleting the $i$th row and $j$th column of $A$. 

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46  
**Chapter 2** Basic Linear Algebra
REVIEW PROBLEMS

Group A

1. Find all solutions to the following linear system:
   \[ x_1 + x_2 = 2 \]
   \[ x_2 + x_3 = 3 \]
   \[ x_1 + 2x_2 + x_3 = 5 \]

2. Find the inverse of the matrix
   \[
   \begin{bmatrix}
   0 & 3 \\
   2 & 1
   \end{bmatrix}
   \]

3. Each year, 20% of all tenured state university faculty become untitled, 5% quit, and 75% remain tenured. Each year, 90% of all tenured State faculty remain tenured and 10% quit. Let \( U_t \) be the number of untitled S.U. faculty at the beginning of year \( t \) and \( T_t \) the tenured number.

   Use matrix multiplication to relate the vector \( \begin{bmatrix} U_{t+1} \\ T_{t+1} \end{bmatrix} \) to the vector \( \begin{bmatrix} U_t \\ T_t \end{bmatrix} \).

4. Use the Gauss-Jordan method to determine all solutions to the following linear system:
   \[ 2x_1 + 3x_2 = 3 \]
   \[ x_1 + x_2 = 1 \]
   \[ x_1 + 2x_2 = 2 \]

5. Find the inverse of the matrix
   \[
   \begin{bmatrix}
   0 & 2 \\
   1 & 3
   \end{bmatrix}
   \]

6. The grades of two students during their last semester at S.U. are shown in Table 2.

   Courses 1 and 2 are four-credit courses, and courses 3 and 4 are three-credit courses. Let GPAi be the semester grade point average for student \( i \). Use matrix multiplication to express the vector \( \begin{bmatrix} GPA_1 \\ GPA_2 \end{bmatrix} \) in terms of the information given in the problem.

7. Use the Gauss-Jordan method to find all solutions to the following linear system:
   \[ 2x_1 + x_2 = 3 \]
   \[ 3x_1 + x_2 = 4 \]
   \[ x_1 - x_2 = 0 \]

8. Find the inverse of the matrix
   \[
   \begin{bmatrix}
   2 & 3 \\
   3 & 5
   \end{bmatrix}
   \]

9. Let \( C_t \) be the number of children in Indiana at the beginning of year \( t \), and \( A_t \) be the number of adults in Indiana at the beginning of year \( t \). During any given year, 5% of all children become adults, and 1% of all children die. Also, during any given year, 3% of all adults die. Use matrix multiplication to express the vector \( \begin{bmatrix} C_{t+1} \\ A_{t+1} \end{bmatrix} \) in terms of \( \begin{bmatrix} C_t \\ A_t \end{bmatrix} \).

10. Use the Gauss-Jordan method to find all solutions to the following linear equation system:
    \[ x_1 = x_3 = 4 \]
    \[ x_2 + x_3 = 2 \]
    \[ x_1 + x_2 = 5 \]

11. Use the Gauss-Jordan method to find the inverse of the matrix
    \[
    \begin{bmatrix}
    1 & 0 & 2 \\
    0 & 1 & 1
    \end{bmatrix}
    \]

12. During any given year, 10% of all rural residents move to the city, and 20% of all city residents move to a rural area (all other people stay put). Let \( R_t \) be the number of rural residents at the beginning of year \( t \), and \( C_t \) be the number of city residents at the beginning of year \( t \). Use matrix multiplication to relate the vector \( \begin{bmatrix} R_{t+1} \\ C_{t+1} \end{bmatrix} \) to the vector \( \begin{bmatrix} R_t \\ C_t \end{bmatrix} \).

13. Determine whether the set \( V = \{[1, 2, 1], [2, 0, 1]\} \) is a linearly independent set of vectors.

14. Determine whether the set \( V = \{[1, 0, 0], [0, 1, 0], [-1, -1, 0]\} \) is a linearly independent set of vectors.

15. Let \( A = \begin{bmatrix}
    a & 0 & 0 & 0 \\
    0 & b & 0 & 0 \\
    0 & 0 & c & 0 \\
    0 & 0 & 0 & d
    \end{bmatrix} \)

   a. For what values of \( a, b, c, \) and \( d \) will \( A^{-1} \) exist?
   b. If \( A^{-1} \) exists, then find it.

16. Show that the following linear system has an infinite number of solutions:
    \[
    \begin{bmatrix}
    1 & 1 & 0 & 0 & | & x_1 \\
    0 & 0 & 1 & 1 & | & x_2 \\
    1 & 0 & 1 & 0 & | & x_3 \\
    0 & 0 & 1 & 3 & | & x_4
    \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix}
    \]

17. Before paying employee bonuses and state and federal taxes, a company earns profits of $60,000. The company pays employees a bonus equal to 5% of after-tax profits. State tax is 5% of profits (after bonuses are paid). Finally, federal tax is 40% of profits (after bonuses and state tax are paid). Determine a linear equation system to find the amounts paid in bonuses, state tax, and federal tax.

18. Find the determinant of the matrix \( A = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \).

19. Show that any \( 2 \times 2 \) matrix \( A \) that does not have an inverse will have det \( A = 0 \).

---

**TABLE 2**

<table>
<thead>
<tr>
<th>Student</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.6</td>
<td>3.8</td>
<td>2.6</td>
<td>3.4</td>
</tr>
<tr>
<td>2</td>
<td>2.7</td>
<td>3.1</td>
<td>2.9</td>
<td>3.6</td>
</tr>
</tbody>
</table>
Group B

20 Let $A$ be an $m \times m$ matrix.
   
   a) Show that if rank $A = m$, then $Ax = 0$ has a unique solution. What is the unique solution?
   
   b) Show that if rank $A < m$, then $Ax = 0$ has an infinite number of solutions.

21 Consider the following linear system:
   
   $$\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} = \begin{bmatrix} [x_1] & [x_2] & \cdots & [x_n] \end{bmatrix} P$$
   
   where
   
   $$P = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ P_{m1} & P_{m2} & \cdots & P_{mn} \end{bmatrix}$$
   
   If the sum of each row of the $P$ matrix equals 1, then use Problem 20 to show that this linear system has an infinite number of solutions.

22 The national economy of Seriland manufactures three products: steel, cars, and machines. (1) To produce $s$ of steel requires $10c$ of steel, $15c$ of cars, and $40m$ of machines. (2) To produce $s$ of cars requires $45c$ of steel, $20c$ of cars, and $100m$ of machines. (3) To produce $s$ of machines requires $40c$ of steel, $10c$ of cars, and $45c$ of machines. During the coming year, Seriland wants to consume $d_s$ dollars of steel, $d_c$ dollars of cars, and $d_m$ dollars of machinery.

   For the coming year, let
   
   $s =$ dollar value of steel produced
   $c =$ dollar value of cars produced
   $m =$ dollar value of machinery produced

   Define $A$ to be the $3 \times 3$ matrix whose $i$th element is the dollar value of product $i$ required to produce $1$ of product $j$ (steel = product 1, cars = product 2, machinery = product 3).

   a) Determine $A$.
   
   b) Show that
   
   $$\begin{bmatrix} s \\ c \\ m \end{bmatrix} = A \begin{bmatrix} s \\ c \\ m \end{bmatrix} + \begin{bmatrix} d_s \\ d_c \\ d_m \end{bmatrix}$$
   
   (Hint: Observe that the value of next year’s steel production = (next year’s consumer steel demand) + (steel needed to make next year’s steel) + (steel needed to make next year’s cars) + (steel needed to make next year’s machines). This should give you the general idea.)

   c) Show that Equation (24) may be rewritten as

   $$(I - A) \begin{bmatrix} s \\ c \\ m \end{bmatrix} = \begin{bmatrix} d_s \\ d_c \\ d_m \end{bmatrix}$$

   d) Given values for $d_s$, $d_c$, and $d_m$, describe how you can use $(I - A)^{-1}$ to determine if Seriland can meet next year’s consumer demand.

   e) Suppose next year’s demand for steel increases by $1$. This will increase the value of the steel, cars, and machines that must be produced next year. In terms of $(I - A)^{-1}$, determine the change in next year’s production requirements.

References

The following references contain more advanced discussions of linear algebra. To understand the theory of linear and nonlinear programming, master at least one of these books:


1Based on Leontief (1966). See references at end of chapter.