Linear Programming

Graphical Solutions.

General Stuff.

Lesson Objectives – The student should understand the following:
   - How to solve a linear programming problem with two variables by graphical methods.
   - Understand the geometrical interpretation of the Simplex procedure.

Graphical Solution of a Linear Program.

When solving a linear program graphically, it is best to represent the problem in the inequality standard form (ISF).

In general, linear programs with \( n \) decision variables (in ISF) have constraints that are half-spaces in \( \mathbb{R}^n \). Thus the types of linear programs suitable for graphical solution are those with two decision variables. The feasible region, which in this case is the intersection of half-planes, can therefore be drawn in \( \mathbb{R}^2 \). Linear programs having three or more variables become much more difficult, even impossible, to visualize.

Example: The Blending Problem (Taha, 1971).

An oil company executive must decide on the best mix of blending processes to implement at one of the company refineries. Process 1 requires five units of Texas crude and three units of North Shore crude. It produces five units of regular gasoline and eight units of high test. Process 2 requires four units of Texas and five units of North Shore to produce four units of regular and four units of high test. There are 200 units of Texas and 150 units of North Shore available. Existing contracts demand at least 100 units of regular and 80 units of high test. Company profits after process expenses are $30K for Process 1 and $40K for Process 2. What processing mix maximizes profits?

We formulate the problem below. Let \( x_1 \) be the number of production runs of Process 1 and \( x_2 \) be the number of runs of Process 2. Both \( x_1 \) and \( x_2 \) must be nonnegative.

Constraints on the Texas and North Shore crudes:

\[
5x_1 + 4x_2 \leq 200 \\
3x_1 + 5x_2 \leq 150
\]
Constraints based on existing contracts are as follows:

\[
\begin{align*}
5x_1 + 4x_2 &\geq 100 \\
8x_1 + 4x_2 &\geq 80
\end{align*}
\]

The objective function is \(30x_1 + 40x_2\) (profit will be given in 1000s of dollars). The complete linear program is written below:

\[
\begin{align*}
\text{max} & \quad 30x_1 + 40x_2 \\
\text{st} & \quad 5x_1 + 4x_2 \leq 200 \\
& \quad 3x_1 + 5x_2 \leq 150 \\
& \quad 5x_1 + 4x_2 \geq 100 \\
& \quad 8x_1 + 4x_2 \geq 80 \\
& \quad x_1 \geq 0, \quad x_2 \geq 0
\end{align*}
\]

The graphical solution: