	x_1	x_2	x_3	x_4	x_5	RHS
$\nabla f(\mathbf{x}_2)$	$-\frac{10}{121}$	$\frac{47}{121}$	0	0	0	—
x_3	0	$\frac{3}{2}$	1	0	$\frac{1}{2}$	11
x_4	0	1	0	1	0	6
x_1	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	7
$\mathbf{r}$	0	$\frac{52}{121}$	0	0	$\frac{5}{121}$	—

When x_1 replaces x_5 in the basis, we get the point $\mathbf{x}_2' = (7, 0, 11, 6, 0)$. Now $\mathbf{q}'\mathbf{x}_2 + \beta = 11$ and $\mathbf{p}'\mathbf{x}_2 + \alpha = -12$, so that, from (11.47), we get $\nabla f(\mathbf{x}_2)' = (-\frac{10}{121}, \frac{47}{121}, 0, 0, 0)$. $\mathbf{B}^{-1}\mathbf{N}$ is given by the columns of x_2 and x_5 in the tableau, and we then get

$$\begin{aligned}\mathbf{r}_N' = (r_2, r_5) &= \nabla_N f(\mathbf{x}_2)' - \nabla_B f(\mathbf{x}_2)'\mathbf{B}^{-1}\mathbf{N}\\ &= (\tfrac{47}{121}, 0) - (0, 0, -\tfrac{10}{121})\begin{bmatrix}\frac{3}{2} & \frac{1}{2}\\ 1 & 0\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}\\ &= (\tfrac{52}{121}, \tfrac{5}{121})\end{aligned}$$

Since $\mathbf{r}_N \geq \mathbf{0}$, we stop with the optimal solution $x_1 = 7$ and $x_2 = 0$. The corresponding objective function value is -1.09.

The Method of Charnes and Cooper [1962]

We now describe another procedure using the simplex method for solving a linear fractional programming problem. Consider the following problem:

$$\begin{aligned}\text{Minimize} \quad & \frac{\mathbf{p}'\mathbf{x} + \alpha}{\mathbf{q}'\mathbf{x} + \beta}\\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b}\\ & \mathbf{x} \geq \mathbf{0}\end{aligned}$$

Suppose that the set $S = \{\mathbf{x}: \mathbf{A}\mathbf{x} \leq \mathbf{b} \text{ and } \mathbf{x} \geq \mathbf{0}\}$ is compact, and suppose that $\mathbf{q}'\mathbf{x} + \beta > 0$ for each $\mathbf{x} \in S$. Letting $z = 1/(\mathbf{q}'\mathbf{x} + \beta)$ and $\mathbf{y} = z\mathbf{x}$, the above problem leads to the following linear program:

$$\begin{aligned}\text{Minimize} \quad & \mathbf{p}'\mathbf{y} + \alpha z\\ \text{subject to} \quad & \mathbf{A}\mathbf{y} - \mathbf{b}z \leq \mathbf{0}\\ & \mathbf{q}'\mathbf{y} + \beta z = 1\\ & \mathbf{y} \geq \mathbf{0}\\ & z \geq 0\end{aligned}$$

First, note that, if $(\mathbf{y}, z)$ is a feasible solution to the above problem, then $z > 0$. This follows since, if $z = 0$, then $\mathbf{y} \neq \mathbf{0}$ must be such that $\mathbf{A}\mathbf{y} \leq \mathbf{0}$ and $\mathbf{y} \geq \mathbf{0}$, which means that $\mathbf{y}$ is a direction of S, violating the compactness assumption. We now demonstrate that, if $(\bar{\mathbf{y}}, \bar{z})$ is an optimal solution to the above linear program, then $\bar{\mathbf{x}} = \bar{\mathbf{y}}/\bar{z}$ is an optimal solution to the fractional program.

Note that $\mathbf{A}\bar{\mathbf{x}} \le \mathbf{b}$ and $\bar{\mathbf{x}} \ge \mathbf{0}$, so that $\bar{\mathbf{x}}$ is a feasible solution to the fractional program. To show optimality of $\bar{\mathbf{x}}$, let $\mathbf{x}$ be such that $\mathbf{A}\mathbf{x} \le \mathbf{b}$ and $\mathbf{x} \ge \mathbf{0}$. Note that $\mathbf{q}'\mathbf{x} + \beta > 0$ by assumption, and that the vector $(\mathbf{y}, z)$ is a feasible solution to the linear program, where $\mathbf{y} = \mathbf{x}/(\mathbf{q}'\mathbf{x} + \beta)$ and $z = 1/(\mathbf{q}'\mathbf{x} + \beta)$. Since $(\bar{\mathbf{y}}, \bar{z})$ is an optimal solution to the linear program, $\mathbf{p}'\bar{\mathbf{y}} + \alpha\bar{z} \le \mathbf{p}'\mathbf{y} + \alpha z$. Substituting for $\bar{\mathbf{y}}$, $\mathbf{y}$, and z, this inequality gives $\bar{z}(\mathbf{p}'\bar{\mathbf{x}} + \alpha) \le (\mathbf{p}'\mathbf{x} + \alpha)/(\mathbf{q}'\mathbf{x} + \beta)$. The result immediately follows by dividing the left-hand side by $1 = \mathbf{q}'\bar{\mathbf{y}} + \beta\bar{z}$.

Now, if $\mathbf{q}'\mathbf{x} + \beta < 0$ for all $\mathbf{x} \in S$, then letting $-z = 1/(\mathbf{q}'\mathbf{x} + \beta)$ and $\mathbf{y} = z\mathbf{x}$ gives the following linear program:

$$
\begin{array}{ll}
\text{Minimize} & -\mathbf{p}'\mathbf{y} - \alpha z \\
\text{subject to} & \mathbf{A}\mathbf{y} - \mathbf{b}z \le \mathbf{0} \\
& -\mathbf{q}'\mathbf{y} - \beta z = 1 \\
& \mathbf{y} \ge \mathbf{0} \\
& z \ge 0
\end{array}
$$

In a fashion similar to that above, if $(\bar{\mathbf{y}}, \bar{z})$ solves the above linear program, then $\bar{\mathbf{x}} = \bar{\mathbf{y}}/\bar{z}$ solves the fractional programming problem.

To summarize, we have shown that a fractional linear program could be solved by a linear programming problem with one additional variable and one additional constraint. The form of the linear program used depends on whether $\mathbf{q}'\mathbf{x} + \beta > 0$ for all $\mathbf{x} \in S$ or $\mathbf{q}'\mathbf{x} + \beta < 0$ for all $\mathbf{x} \in S$. If there exist $\mathbf{x}_1, \mathbf{x}_2 \in S$ such that $\mathbf{q}'\mathbf{x}_1 + \beta > 0$ and $\mathbf{q}'\mathbf{x}_2 + \beta < 0$, then the optimal solution to the fractional program is unbounded.

11.4.4 Example

Consider the following problem:

$$
\begin{array}{ll}
\text{Minimize} & \dfrac{-2x_1 + x_2 + 2}{x_1 + 3x_2 + 4} \\
\text{subject to} & -x_1 + x_2 \le 4 \\
& 2x_1 + x_2 \le 14 \\
& x_2 \le 6 \\
& x_1, x_2 \ge 0
\end{array}
$$

The feasible region for this problem is shown in Figure 11.5. We solve this problem using the method of Charnes and Cooper. Note that the point (0, 0) is feasible and that at this point $-x_1 + 3x_2 + 4 > 0$. Hence, the denominator is positive over the entire feasible region. The equivalent linear program is given by:

$$
\begin{array}{ll}
\text{Minimize} & -2y_1 + y_2 + 2z \\
\text{subject to} & -y_1 + y_2 - 4z \le 0 \\
& 2y_1 + y_2 - 14z \le 0 \\
& y_2 - 6z \le 0 \\
& y_1 + 3y_2 + 4z = 1 \\
& y_1, y_2, z \ge 0
\end{array}
$$

The reader can verify that $y_1 = \frac{7}{11}$, $y_2 = 0$, and $z = \frac{1}{11}$ is an optimal solution to the above linear program. Hence, the optimal solution to the original problem is $x_1 = y_1/z_1 = 7$ and $x_2 = y_2/z_2 = 0$.

11.31 In Section 11.3, we approximated a separable programming problem using the λ-form. An alternative, called the δ-*form*, was considered in Exercise 11.29. Consider a variable x in the interval $[a, b]$ and grid points $\mu_1 = a, \mu_2, \ldots, \mu_k = b$. Then, using the λ- and δ-forms, x can be represented as

1. $x = \sum_{j=1}^{k} \lambda_j \mu_j \qquad \sum_{j=1}^{k} \lambda_j = 1 \qquad \lambda_j \geq 0 \qquad \text{for } j = 1, \ldots, k$
   where $\lambda_p \lambda_q = 0$ if μ_p and μ_q are not adjacent.
2. $x = \mu_1 + \sum_{j=1}^{k-1} \Delta_j \delta_j \qquad 0 \leq \delta_j \leq 1 \qquad \text{for } j = 1, \ldots, k-1$
   $\delta_i > 0 \Rightarrow \delta_j = 1 \qquad \text{for } j < i$

Show that the two forms are related by the relationship

$$\lambda_j = \begin{cases} \delta_{j-1} - \delta_j & \text{if } j = 1, \ldots, k-1 \\ \delta_{j-1} & \text{if } j = k \end{cases}$$

where $\delta_0 = 1$. In particular, show that this relationship could be written in vector form as $\boldsymbol{\lambda} = \mathbf{T}\boldsymbol{\delta}$, where $\mathbf{T}$ is an upper triangular matrix.

11.32 Solve the following problem by the two linear fractional programming algorithms discussed in Section 11.4:

$$\begin{aligned} \text{Minimize} \quad & \frac{-2x_1 + 3x_2 + 5x_3 + 2}{x_1 + 2x_2 + x_3 + 2} \\ \text{subject to} \quad & 2x_1 + 3x_2 + x_3 \leq 12 \\ & x_1 \quad 2x_2 \quad \geq 2 \\ & x_1 \quad + x_3 \leq 8 \\ & x_1, \quad x_2, \quad x_3 \geq 0 \end{aligned}$$

11.33 Consider the following problem:

$$\begin{aligned} \text{Maximize} \quad & \frac{8x_1 + 6x_2 - 5}{-4x_1 + 2x_2 - 40} \\ \text{subject to} \quad & x_1 + x_2 \leq 10 \\ & 3x_1 - 5x_2 \leq 6 \\ & x_1, x_2 \geq 0 \end{aligned}$$

a. Solve the problem by the method of Gilmore and Gomory.
b. Solve the problem by the method of Charnes and Cooper.

11.34 Suppose that the region $\{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ is unbounded. Further, suppose that an improving feasible direction $\mathbf{d}$ is found while minimizing a linear fractional function over the above region. In particular, suppose that $\mathbf{d}_N$ consists of a vector of zeros,.except for a 1 at position j and $\mathbf{d}_B = -\mathbf{B}^{-1}\mathbf{a}_j \geq \mathbf{0}$. Is it necessarily true that the optimal objective is unbounded by moving from the current extreme point in the direction $\mathbf{d}$? If not, what are the possible cases that can be encountered?

11.35 Consider the function f defined by

$$f(\mathbf{x}) = \frac{x_1 + 2x_2 - 6}{3x_1 - x_2 + 2}$$

a. Sketch the following sets in the (x_1, x_2) plane, and determine whether they are convex:
   $S = \{(x_1, x_2) : f(\mathbf{x}) \leq 2\}$
   $S_1 = S \cap \{(x_1, x_2) : 3x_1 - x_2 + 2 > 0\}$
   $S_2 = S \cap \{(x_1, x_2) : 3x_1 - x_2 + 2 < 0\}$
b. Is your conclusion in part a inconsistent with the fact that f is quasiconvex on the region $\{(x_1, x_2) : 3x_1 - x_2 + 2 \neq 0\}$? Discuss.

11.36 Let

$$f(\mathbf{x}) = \frac{\mathbf{p}^t\mathbf{x} + \alpha}{\mathbf{q}^t\mathbf{x} + \beta}$$

and let $S = \{\mathbf{x} : \mathbf{q}^t\mathbf{x} + \beta > 0\}$. Show directly that f is quasiconvex, quasiconcave, strictly quasiconvex, and strictly quasiconcave on S.

11.37 Let $f : E_n \rightarrow E_1$ be quasiconcave, and let $\theta(\lambda) = f(\mathbf{x} + \lambda\mathbf{d})$, where $\mathbf{x}$ is a given vector and $\mathbf{d}$ is a given direction.

a. Show that θ is a quasiconcave function in λ.
b. Consider the problem to minimize $\theta(\lambda)$ subject to $\lambda \in [a, b]$. Show that, if $\nabla f(\mathbf{x})^t \mathbf{d} < 0$, then $\lambda = b$ is an optimal solution to the above problem.
c. Letting $f(\mathbf{x}) = (\mathbf{p}^t\mathbf{x} + \alpha)/(\mathbf{q}^t\mathbf{x} + \beta)$, use the result in part b to show that no line search is needed for solving linear fractional programs by the convex–simplex method.

11.38 In solving a linear fractional programming problem, suppose that we add the following two rows to the initial tableau:

$$z_1 - \mathbf{p}^t\mathbf{x} = \alpha$$
$$z_2 - \mathbf{q}^t\mathbf{x} = \beta$$

As the problem is solved by the convex–simplex method, the coefficients of the basic vector $\mathbf{x}_B$ in these rows are equal to zero so that the updated rows are given by

$$z_1 - (\mathbf{p}_N^t - \mathbf{p}_B^t\mathbf{B}^{-1}\mathbf{N})\mathbf{x}_N = \alpha + \mathbf{p}_B^t\mathbf{B}^{-1}\mathbf{b}$$
$$z_2 - (\mathbf{q}_N^t - \mathbf{q}_B^t\mathbf{B}^{-1}\mathbf{N})\mathbf{x}_N = \beta + \mathbf{q}_B^t\mathbf{B}^{-1}\mathbf{b}$$

Show that the reduced gradient vector $\mathbf{r}_N$ is given by

$$\mathbf{r}_N^t = \frac{(\mathbf{p}_N^t - \mathbf{p}_B^t\mathbf{B}^{-1}\mathbf{N})\bar{z}_2 - (\mathbf{q}_N^t - \mathbf{q}_B^t\mathbf{B}^{-1}\mathbf{N})\bar{z}_1}{\bar{z}_2^2}$$

where $\bar{z}_1 = \alpha + \mathbf{p}_B^t\mathbf{B}^{-1}\mathbf{b}$ and $\bar{z}_2 = \beta + \mathbf{q}_B^t\mathbf{B}^{-1}\mathbf{b}$. Note that each term in the expression for $\mathbf{r}_N$ is immediately available from the updated tableau. Solve the problem in Example 11.4.3 using the above procedure for computing $\mathbf{r}_N$.

11.39 Verify that the separable objective function (11.62a) of the dual geometric program (DGP) is concave.

11.40 Consider the geometric programming problem of Example 11.5.4, and let C denote the ratio of the cost per unit length (cm) of the wire to the cost per unit surface area (cm^2) of the cylinder. Analyze this problem to study the sensitivity of the optimal dimensions of the cylinder to this cost factor C.

11.41 Consider the problem to minimize $f_1(\mathbf{x}) + [f_2(\mathbf{x})]^a f_3(\mathbf{x})$ where f_i, $i = 1, 2, 3$ are posynomials and where $a > 0$. Show that this is equivalent to the standard posynomial geometric program GP to minimize $f_1(\mathbf{x}) + x_0^a f_3(\mathbf{x})$ subject to $x_0^{-1} f_2(\mathbf{x}) \le 1$, where x_0 is an additional variable. Illustrate by solving the problem to minimize $x_1^{-1/2}x_2^{1/8} + [\frac{4}{5}x_1^{1/2}x_2^{2/3} + \frac{2}{5}x_1^{1/3}x_2]^{1/2}x_1^{1/4}x_2^{-1/2}$.

11.42 Consider the problem to minimize

$$f_1(\mathbf{x}) + \frac{f_2(\mathbf{x})}{[f_3(\mathbf{x}) - f_4(\mathbf{x})]^a}$$

where f_i, $i = 1, \ldots, 4$, are posynomials but f_3 has only one term, and where $a > 0$. Show that this is equivalent to the standard posynomial geometric program to minimize $f_1(\mathbf{x}) + f_2(\mathbf{x})x_0^{-a}$ subject to $x_0/f_3(\mathbf{x}) + f_4(\mathbf{x})/f_3(\mathbf{x}) \le 1$.

11.43 Consider the problem to minimize $f_1(\mathbf{x}) - f_2(\mathbf{x})$, where f_1 and f_2 are posynomials, f_2 has only one term, and the optimal value is known to be negative. Show that this can be equivalently solved as the standard posynomial geometric program to minimize x_0^{-1} subject to $[x_0/f_2(\mathbf{x})] + [f_1(\mathbf{x})/f_2(\mathbf{x})] \le 1$.

11.44 Suppose that a metal wire frame has to be constructed for a rectangular box having a skeleton and dimensions (in cm) as shown below.