Practice 3 Solutions

1. There are two directions to consider

"⇒" Let \( x^* \) be opt for (P1). Then
\[
C^T x^* \geq c^T x \forall \text{ feasible } x \text{ and } x^* \text{ satisfies } A x^* = b, x^* \geq 0.
\]
Thus \( x^* \) is feasible for (P2). Furthermore
\[
C^T x^* \geq c^T x \Rightarrow \alpha(c^T x^*) \geq \alpha(c^T x) \Rightarrow \\
\alpha(c^T x^*) + \beta \geq \alpha(c^T x) + \beta \forall \text{ feasible } x
\]
Thus \( x^* \) is optimal for (P2).

"⇐" Let \( x^* \) be opt for (P2). Then \( x^* \) satisfies
\[
A x^* = b, x^* \geq 0.
\]
Thus \( x^* \) is feasible for (P1). Since \( x^* \) is opt for (P2)
\[
\alpha(c^T x^*) + \beta \geq \alpha(c^T x) + \beta \forall \text{ feasible } x
\]
\[
\Rightarrow \alpha(c^T x^*) \geq \alpha(c^T x)
\]
\[
C^T x^* \geq c^T x
\]
Thus \( x^* \) is optimal for (P1).
You are given the following information about a project consisting of six activities:

<table>
<thead>
<tr>
<th>Activity</th>
<th>Predecessors</th>
<th>Duration (mins)</th>
</tr>
</thead>
<tbody>
<tr>
<td>START</td>
<td>None</td>
<td>0</td>
</tr>
<tr>
<td>A</td>
<td>START</td>
<td>5</td>
</tr>
<tr>
<td>B</td>
<td>START</td>
<td>1</td>
</tr>
<tr>
<td>C</td>
<td>B</td>
<td>2</td>
</tr>
<tr>
<td>D</td>
<td>A, C</td>
<td>4</td>
</tr>
<tr>
<td>E</td>
<td>A</td>
<td>6</td>
</tr>
<tr>
<td>F</td>
<td>D, E</td>
<td>3</td>
</tr>
<tr>
<td>FINISH</td>
<td>F</td>
<td>0</td>
</tr>
</tbody>
</table>

(a) Formulate a linear program to determine the start times for each activity that will minimize the project’s duration. (Do NOT use generic modeling.)

\[
\begin{align*}
X_j &= \text{start time activity } j \\
X_A - X_{\text{START}} &\geq 0 \\
X_B - X_{\text{START}} &\geq 0 \\
X_C - X_B &\geq 1 \\
X_D - X_A &\geq 5 \\
X_D - X_C &\geq 2 \\
X_E - X_A &\geq 5 \\
X_E - X_D &\geq 4 \\
X_F - X_D &\geq 6 \\
X_{\text{FINISH}} - X_F &\geq 3 \\
x_j &= \text{U.S.} \\
\min X_{\text{FINISH}} - X_{\text{START}} &= \text{objective is to}
\min \text{project time}
\end{align*}
\]

\[\text{ES}(j) \geq \text{ES}(i) + d_i \]

where activity \( i \) precedes activity \( j \), \( d_i \) is duration of activity \( i \)
(b) Find the dual for the LP you created in part (a).

\[ \text{max} \quad y_{BC} + 5y_{AD} + 2y_{CD} + 5y_{AE} + 4y_{DF} + 6y_{EF} + 3y_{F,FINISH} \]

\[ \text{s. t.} \quad -y_{START, A} - y_{START, B} = -1 \]
\[ y_{START, A} - y_{AD} - y_{AE} = 0 \]
\[ y_{START, B} - y_{BC} = 0 \]
\[ y_{BC} - y_{CD} = 0 \]
\[ y_{AD} + y_{CD} - y_{DF} = 0 \]
\[ y_{AE} - y_{EF} = 0 \]
\[ y_{DF} + y_{EF} - y_{F,FINISH} = 0 \]
\[ y_{F,FINISH} = 1 \]
\[ y_{ij} \geq 0 \quad \forall (i, j) \]
3. Let \( y_i = \{ \begin{cases} 1 & \text{if option } i \text{ is used} \\ 0 & \text{otherwise} \end{cases} \) \( i = 1, \ldots, 6 \)

\[
\text{max } 1y_1 + 0.2y_2 + 3y_3 + 4y_4 + 0.45y_5 + 0.45y_6
\]

(customers reached in millions)

S.T.

\[
0.5y_1 + 0.15y_2 + 0.3y_3 + 0.25y_4 + 0.25y_5 + 0.1y_6 \leq 1.8 \quad \text{(budget)}
\]

\[
y_1 + 2.5y_2 + 2y_3 + 2y_4 + 2y_5 + 4y_6 \leq 15 \quad \text{(design hrs)}
\]

\[
2y_1 + 1y_2 + 1y_3 + 1y_4 + 1y_5 + 10y_6 \leq 12 \quad \text{(sales hrs)}
\]

\[
y_6 \leq y_4 + y_5 \quad \text{(promo camp - radio)}
\]

\[
y_2 + y_5 \leq 5 \quad \text{(trade - pop mags)}
\]

\[
y_1 + y_2 + y_3 + y_4 + y_5 + y_6 \leq 3 \quad \text{(at most 3)}
\]

\[
y_1 + y_5 + y_6 \leq 2 \quad \text{(pop mag - promo - tv)}
\]

\[
y_i \in \{0, 1\} \quad \forall i
\]

\[
4. \quad \text{(a)}
\]
### Feas Point | \text{OF}(x_1 + 5x_2)
--- | ---
(0,0) | 0
(1,0) | 1
(2,0) | 2
(3,0) | 3
(4,0) | 4
(5,0) | 5
(6,0) | 6
(0,1) | 5
(1,1) | 6
(2,1) | 7
(3,1) | 8
(4,1) | 9
(5,1) | 10
(0,2) | 11
(1,2) | 12
(2,2) | 13
(3,2) | 14
(4,2) | 16
(1,3) | 17
(2,3) | 18
(3,3) | 19
(4,3) | 22
(2,4) | 23
(3,4) | opt soln

(b) \( L_0 : \max Z = x_1 + 5x_2 \)
\[ \text{s.t.} \ -4x_1 + 3x_2 \leq 6 \]
\[ 3x_1 + 2x_2 \leq 18 \]
\[ x_1, x_2 \geq 0 \]

\( x^* = (42/7, 90/17) \)
\[ z^* = \frac{492}{17} \approx 28.94 \]

\[ L_1: L_0 + x_1 \geq 3 \quad x^* = (3, 9/2) \quad z^* = 25\sqrt{2} \]

\[ L_2: L_0 + x_1 \leq 2 \quad x^* = (2, 14/3) \quad z^* = 25\sqrt{3} \]

\[ L_3: L_1 + x_2 \geq 5 \quad \text{unfeas} \]

\[ L_4: L_1 + x_2 \leq 4 \quad x^* = (10/5, 4) \quad z^* = 23\sqrt{3} \]

\[ L_5: L_4 + x_1 \leq 3 \quad x^* = (3, 4) \quad z^* = 23 \]

\[ L_6: L_4 + x_1 \geq 4 \quad x^* = (4, 3) \quad z^* = 19 \]
\[ L_1: L_2 + x_2 \geq 5 \text{ infeas} \]

\[ L_2: L_2 + x_2 \leq 4 \quad x^* = (2, 4) \quad z^* = 22 \]

\[
\begin{array}{c|c|c}
L_0 & u_B = 28.94 & L_B = 0 \\
\hline
x_1 = 2.47 & x_2 = 5.29 & z = 28.94 \\
\end{array}
\]

\[ x_1 \geq 3 \]

\[
\begin{array}{c|c|c|c|c}
L_1 & x_1 = 3 & x_2 = 4.5 & z = 25.5 \\
\hline
L_3 & \text{infeas} & & \\
\hline
L_4 & x_1 = 3\sqrt{3} & x_2 = 4 & z = 23 \sqrt{3} \\
\end{array}
\]

\[ x_4 \leq 4 \]

\[ x_4 \geq 5 \]

\[
\begin{array}{c|c|c|c|c}
L_5 & x_1 = 3 & x_2 = 4 & z = 23 \\
\hline
L_6 & x_1 = 4 & x_2 = 3 & z = 19 \\
\end{array}
\]

\[ x_1 \leq 2 \]

\[
\begin{array}{c|c|c|c|c}
L_2 & x_1 = 2 & x_2 = 4 \sqrt{3} & z = 25 \sqrt{3} \\
\hline
L_7 & \text{infeas} & & \\
\end{array}
\]

\[ x_2 \geq 5 \]

\[
\begin{array}{c|c|c|c|c}
L_8 & x_1 = 2 & x_2 = 4 & z = 22 \\
\end{array}
\]

Opt soln \((3, 4)\) \(z = 23\)

Note: we could have first branched on \(x_2\) \((x_2 \leq 5, x_2 \geq 6)\)
we would have obtained a different Branch & Bound tree.
This is a maximum assignment problem.
we have \# people = \# jobs = 4
\[ C = \max \{c_{ij}\} = 82 \] replace \( c_{ij} \) w/ \( C - c_{ij} \)
so we have a minimum assignment problem

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>Row Min</th>
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<tbody>
<tr>
<td>Clyde</td>
<td>20</td>
<td>17</td>
<td>2</td>
<td>32</td>
<td>2</td>
</tr>
<tr>
<td>Carlos</td>
<td>6</td>
<td>12</td>
<td>0</td>
<td>27</td>
<td>0</td>
</tr>
<tr>
<td>Sid</td>
<td>7</td>
<td>42</td>
<td>5</td>
<td>25</td>
<td>5</td>
</tr>
<tr>
<td>Ted</td>
<td>37</td>
<td>34</td>
<td>32</td>
<td>46</td>
<td>32</td>
</tr>
</tbody>
</table>

subtract row mins

<table>
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<th>2</th>
<th>3</th>
<th>4</th>
</tr>
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<tbody>
<tr>
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<td>15</td>
<td>0</td>
<td>30</td>
</tr>
<tr>
<td>Carlos</td>
<td>6</td>
<td>12</td>
<td>0</td>
<td>27</td>
</tr>
<tr>
<td>Sid</td>
<td>2</td>
<td>37</td>
<td>0</td>
<td>20</td>
</tr>
<tr>
<td>Ted</td>
<td>5</td>
<td>2</td>
<td>0</td>
<td>14</td>
</tr>
</tbody>
</table>

subtract col mins

cover all zeros w/ lines
smallest uncovered is 6
<table>
<thead>
<tr>
<th>Clyde</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Carlos</td>
<td>4</td>
<td>4</td>
<td>0</td>
<td>7</td>
</tr>
<tr>
<td>Std</td>
<td>0</td>
<td>29</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Ted</td>
<td>9</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Assign Clyde to FSU effectiveness: 80
Carlos: UT
Sid: CSUF
Ted: ASU

\[
\begin{align*}
\text{Total} & = 261 \\
\end{align*}
\]

\[
\begin{align*}
\text{Total} & = 261 \\
\end{align*}
\]