1. Let $f(x, y) = x^2 - y^2$.

(a) **(8 pts)** Compute f_{xx} and f_{yy}.

$f_x = 2x$ $\qquad$ $f_y = -2y$

$f_{xx} = 2$ $\qquad$ $f_{yy} = -2$

(b) **(2 pts)** Show that $f_{xx} + f_{yy}$ is always 0.

$f_{xx} + f_{yy} = 2 + (-2) = 0$

2. Consider the function $f(x, y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$.

(a) **(12 pts)** Find all critical points of $f(x, y)$. (Hint : There are four)

$f_x = 3x^2 + 3y^2 - 6x$

$f_y = 6xy - 6y$

No critical points where f_x OR f_y is undefined.

So we solve.

$f_x = 0 \quad , \quad f_y = 0$

$\Rightarrow \quad 3x^2 + 3y^2 - 6x = 0$

$\quad \& \quad 6xy - 6y = 0$

The second equation is $6y(x-1) = 0$

$\Rightarrow$ either $y = 0$ OR $x = 1$

Case 1: If $y = 0$, the first equation becomes.

$3x^2 + 3(0)^2 - 6x$

$\Rightarrow 3x^2 - 6x = 0 \quad \Rightarrow \quad x = 0, 2$

Case 2: If $x = 1$, the first eq. becomes

$3(1)^2 + 3y^2 - 6(1) = 0$

$\Rightarrow y = 1, -1$

So critical pts: $\boxed{y=0,\ x=0}$ $\boxed{y=0, x=2}$ $\boxed{x=1, y=1}$ $\boxed{x=1, y=-1}$

(b) **(8 pts)** Decide whether each critical point found in (a) is a relative minimum, relative maximum, saddle point or indeterminable.

$f_{xx} = 6x - 6 \qquad f_{yy} = 6x - 6$

$f_{xy} = 6y$

First critical pt. $\boxed{y=0,\ x=0}$

$f_{xx} = -6, \quad f_{yy} = -6, \quad f_{xy} = 0$

$d = f_{xx} f_{yy} - (f_{xy})^2 = 36 - 0^2 = 36$

$d > 0, \quad f_{xx} < 0 \implies$ relative maximum

2nd critical pt. $\boxed{y=0,\ x=2}$

$f_{xx} = +6 \qquad f_{yy} = +6$

$f_{xy} = 0$

$d = f_{xx} f_{yy} - (f_{xy})^2 = +36.$

$d > 0, \quad f_{xx} > 0 \implies$ relative minimum ~~saddle point~~

3rd critical pt. $\boxed{x=1,\ y=1}$ $\quad f_{xx} = 0, \ f_{yy} = 0, \ f_{xy} = 6$

$d = -36 < 0 \implies$ saddle pt.

4th critical pt. $\boxed{x=1,\ y=-1}$ $\quad f_{xx} = 0 = f_{yy}, \ f_{xy} = -6$

$d = 0 \cdot 0 - (-6)^2 = -36 < 0$

$\implies$ saddle point

3. **(15 pts)** The airline industry puts a limit on the size of the checked baggage on any flight. The restriction states that the SUM of the length, breadth and height of the suitcase can be at most 300 inches. Find the dimensions of the suitcase satisfying this restriction that has the MAXIMUM volume. Show that it is in fact a cube.

Let length $= l$, ~~b~~breadth $= b$, height $= h$.

Volume $= lbh$.

Constraint says $l + b + h = 300$

Set up problem: $\max \; lbh$

~~s~~ subject to

$l + b + h - 300 = 0$

Using Lagrange Multipliers:

$$F(l, b, h, \lambda) = lbh - \lambda(l + b + h - 300)$$

$$\frac{\partial F}{\partial l} = bh - \lambda = 0 \qquad (1)$$

$$\frac{\partial F}{\partial b} = lh - \lambda = 0 \qquad (2)$$

$$\frac{\partial F}{\partial h} = lb - \lambda = 0 \qquad (3)$$

$$\frac{\partial F}{\partial \lambda} = -l - b - h + 300 = 0 \qquad (4)$$

From first 2 equations $bh = \lambda$, $lh = \lambda \Rightarrow bh = lh$

$\Rightarrow b = l$

or $h = 0$

$h = 0$ ~~m~~means volume is 0, ~~so~~ clearly not maximum. So $\underline{l = b}$.

From eqs. (2) + (3) we get $\underline{h = b}$. So $l = b = h$.

So eq. (4) becomes:

$$-l - b - h + 300 = 0$$

$$\Rightarrow -3l + 300 = 0 \quad \Rightarrow \quad l = 100$$

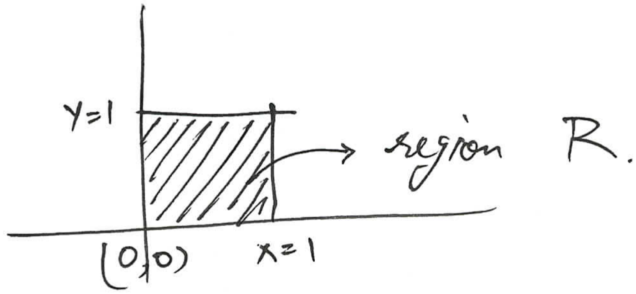
So box with max. volume

has length = breadth = height

= 100 inches.

and volume = 100^3 inches^3.

4. **(10 pts)** Compute $\iint_R xy\,dA$ where R is the square region given by $0 \le x \le 1$ and $0 \le y \le 1$.



$$\iint_R xy\,dA = \int_0^1\int_0^1 xy\,dx\,dy$$

$$= \int_0^1 \left[\frac{x^2y}{2}\Big|_0^1 \right] dy$$

$$= \int_0^1 \frac{y}{2}\,dy = \frac{y^2}{4}\Big|_0^1$$

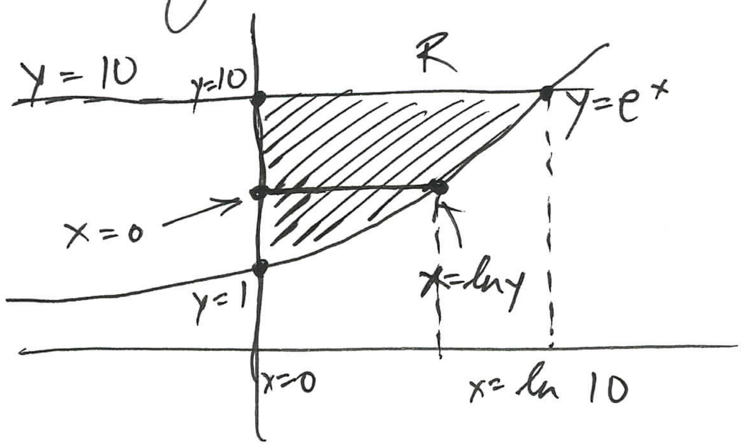
$$= \boxed{\frac{1}{4}}$$

5. **(15 pts)** Compute $\int_0^{\ln 10}\int_{e^x}^{10} \frac{1}{\ln y} dy dx$. (Hint : You may consider changing the order of integration)

Don't know how to integrate $\int \frac{1}{\ln y} dy$

So change order of integration.

1. Sketch region R over which integral is computed.



Inner limits for y

$e^x - 10$

So sketch

$y=e^x$, $y=10$

Outer limits are

$x=0$ to $x=\ln 10$

2. Choose order $\int\int \frac{1}{\ln y} dx\, dy$

Need Inner limits. y is fixed & want limits on x.

from picture: $x=0$ – $x=\ln y$

Outer limits y goes from 1 to 10.

So :

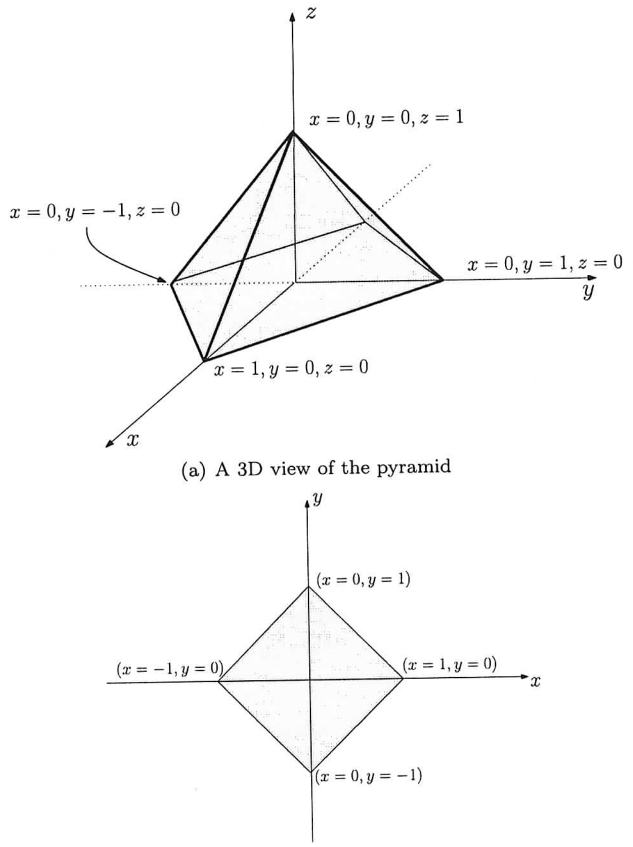
$$\int_1^{10}\int_0^{\ln y}\left(\frac{1}{\ln y}\right) dx\, dy = \int_1^{10}\left[\frac{x}{\ln y}\right]_0^{\ln y} dy = \int_1^{10}(1)dy$$

$$= y\Big|_1^{10} = \boxed{9}$$

6. You will compute the volume of a pyramid using double integrals. Consider the pyramid whose base is on the xy plane and the four slanted sides are given by the following four planes :

$$\begin{aligned} z &= 1-x-y \\ z &= 1-x+y \\ z &= 1+x-y \\ z &= 1+x+y \end{aligned}$$

Note that this means the top vertex of the pyramid is the point $(x = 0, y = 0, z = 1)$; all the four planes meet at that point. Figure 1(a) shows a 3D view of the pyramid and Figure 1(b) shows the base drawn on the xy plane.



(a) A 3D view of the pyramid

(b) A drawing of the base of the pyramid

Figure 1: Diagrams for Problem 6

(a) **(8 pts)** Determine the equations of the four lines forming the sides of the base in the xy plane.

$$x+y=1 \qquad x-y=1$$

$$-x+y=1 \qquad -x-y=1$$

These are the ~~traces~~ xy traces of the four planes forming the sides of the pyramid.

(b) **(2 pts)** Observe that the pyramid is symmetrical about the z axis and its volume is four times the volume of the section of the pyramid lying in the first octant. (Recall that the first octant is the region in space satisfying $x \geq 0, y \geq 0, z \geq 0$). This section of the pyramid is the part of the first octant bounded by one of the four planes given above. Write the equation of this plane.

$$z = 1-x-y$$

(c) **(5 pts)** Use the observations made in (b) to set up (but DO NOT EVALUATE) the double integral to compute the volume of the pyramid.

The region R for the double integral is the triangle $(0,0)$, $(1,0)$, $(0,1)$

This double integral computes the volume of the part of the pyramid in the first octant.

so volume of pyramid $= 4\int_0^1\int_0^{1-y}(1-x-y)\,dx\,dy$

7. Write the n-th term in the following sequences :

(a) **(2 pts)** $-\frac{1}{2}, \frac{4}{3}, -\frac{9}{4}, \frac{16}{5}, \ldots$

Not in syllabus. !!

(b) **(2 pts)** $2, 2, \frac{8}{6}, \frac{16}{24}, \frac{32}{120}, \ldots$

(c) **(2 pts)** $1, \frac{5}{8}, \frac{7}{15}, \frac{9}{24}, \frac{11}{35}, \frac{13}{48}, \ldots$

8. Determine if the following sequences converge or diverge. If the sequence converges, write the limit.

(a) **(3 pts)** $a_n = \frac{2^n+1}{3 \cdot 2^n}$

(b) **(3 pts)** $a_n = (-1)^n(\frac{1}{n^2+3})$

(c) **(3 pts)** $a_n = (-1)^n(\frac{n+1}{n})$