1. Find the following limits.

(a) **(4 pts)** (Hint : Simplify fractions first)

$$\lim_{x \to 2} \frac{\frac{1}{x} - \frac{1}{2}}{x - 2}.$$

$$= \lim_{x \to 2} \frac{\left(\frac{2-x}{2x}\right)}{x-2} = \lim_{x \to 2} -\frac{1}{2x}$$

$$= \boxed{-\frac{1}{4}}$$

(b) **(4 pts)** (Hint : Rationalize the numerator and divide out)

$$\lim_{x \to 1} \frac{\sqrt{x} - 1}{x^2 - 1}.$$

$$= \lim_{x \to 1} \frac{(\sqrt{x} - 1)}{x^2 - 1} \frac{(\sqrt{x} + 1)}{(\sqrt{x} + 1)}$$

$$= \lim_{x \to 1} \frac{(\sqrt{x})^2 - (1)^2}{(x^2-1)(\sqrt{x}+1)} = \lim_{x \to 1} \frac{x-1}{(x^2-1)(\sqrt{x}+1)}$$

$$= \lim_{x \to 1} \frac{\cancel{(x-1)}}{(x+1)\cancel{(x-1)}(\sqrt{x}+1)} = \lim_{x \to 1} \frac{1}{(x+1)(\sqrt{x}+1)}$$

$$= \frac{1}{(1+1)(\sqrt{1}+1)} = \boxed{\frac{1}{4}}$$

(c) **(4 pts)**

$$\lim_{x\to\infty} \frac{5x^3 - 3x}{21 + 3x^2 - 6x^3}.$$

$$= \boxed{\frac{5}{-6}} = \boxed{-\frac{5}{6}}$$

both are OK!

2. Differentiate the following functions. DO NOT SIMPLIFY ANSWERS.

(a) **(7 pts)**

$$y = \frac{\sqrt{5+2x}}{\cos(3x+1)}$$

$$\frac{dy}{dx} = \frac{\left[\frac{1}{2}(5+2x)^{-1/2}(2)\right]\cos(3x+1) - \sqrt{5+2x}\left(-\sin(3x+1)(3)\right)}{\left(\cos(3x+1)\right)^2}$$

$$= \boxed{\frac{\frac{\cos(3x+1)}{(5+2x)^{1/2}} + 3\sqrt{5+2x}\left(\sin(3x+1)\right)}{\left(\cos(3x+1)\right)^2}}$$

(b) **(6 pts)**

$$y = (\tan(\sin(x)))^3$$

<u>Use Chain Rule</u>

$$\frac{dy}{dx} = 3(\tan(\sin(x)))^2 \sec^2(\sin(x)) \cos(x).$$

3. **(5 pts)** Solve the equation for all values of θ in the interval $[0, 2\pi]$:

$$\cos(\theta/2) - \cos(\theta) = 1.$$

$$\cos\frac{\theta}{2} = \frac{1+\cos\theta}{2}$$

$\therefore$ we get $\frac{1+\cos\theta}{2} - \cos\theta = 1$

Multiply both sides by 2 :

$\Rightarrow$ $1 + \cos\theta - 2\cos\theta = 2$

$\Rightarrow$ $1 - \cos\theta = 2.$

$\Rightarrow$ $\cos\theta = -1$

$\Rightarrow$ $\boxed{\theta = \pi}$

4. Consider the function

$$f(x) = \frac{6x^2}{x-6}.$$

(This question has three parts a), b) and c) - all three parts refer to this function.)

(a) **(5 pts)** Determine all the intervals where the function is increasing and decreasing.

Final answer:

Increasing on $(-\infty, 0), (12, \infty)$

Decreasing on $(0,6), (6,12)$ or $(0,12)$

both are OK

$$f'(x) = \frac{12x(x-6) - 6x^2}{(x-6)^2} = \frac{6x^2 - 6(12x)}{(x-6)^2}$$

$$= \frac{6x(x-12)}{(x-6)^2}$$

Critical points : $x = 0, 6, 12.$

$+ \quad - \quad - \quad +$

$-1 \quad 0 \quad 1 \quad 6 \quad 7 \quad 12 \quad 13$

At $x=-1$: $f'(-1) = \frac{6(-1)(-1-12)}{(-1-6)^2} = \frac{6(13)}{(-7)^2} > 0$

At $x=1$, $f'(1) = \frac{6(1)(1-12)}{(1-6)^2} < 0$, At $x=7$, $f'(7) = \frac{6(7)(7-12)}{(7-6)^2} < 0$

At $x=13$, $f'(13) = \frac{6(13)(13-12)}{(13-6)^2} > 0$

(b) **(5pts)** Find the relative minimum and relative maximum of $f(x)$. You need to say at what values of x these relative extrema occur, and identify whether it is a relative minimum or relative maximum. You DO NOT need to report the value of the function at these points.

Looking at the signs from part a)

$x=0$ is a relative maximum

$x=12$ is a relative minimum

(c) **(5pts)** Find the absolute maximum and absolute minimum values of the function in the interval $-1 \leq x \leq 2$.

Critical points in this interval : $x=0$.

So check $f(-1)$, $f(0)$, $f(2)$.

$f(-1) = \dfrac{6(-1)^2}{-1-6} = \dfrac{6}{-7}$, $f(0) = 0$, $f(2) = \dfrac{6(2)^2}{2-6} = -6$

$\boxed{\text{Absolute maximum} = 0, \quad \text{Absolute minimum} = -6}$

5. **(10 pts)** Consider the function

$$x(x-3)^2.$$

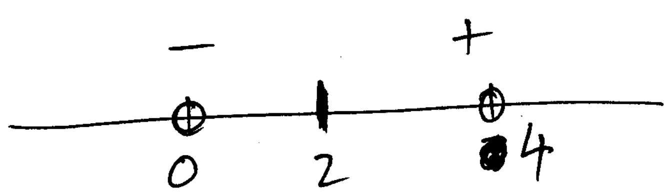
Determine the intervals where the function is concave upwards and concave downwards. Also indicate the points of inflection.

$f'(x) = (x-3)^2 + x(2(x-3))$

$= (x-3)^2 + 2x(x-3) = (x-3)^2 + 2x^2 - 6x$

$f''(x) = 2(x-3) + 4x - 6 = 6x - 12$

Critical points: $6x - 12 = 0 \Rightarrow x = \frac{12}{6} = 2$



At $x=0$, $f''(0) = -12 < 0$

At $x=3$, $f''(0) = 6 > 0$

Concave downwards over $(-\infty, 2)$

Concave upwards over $(2, \infty)$

$x=2$ is a pt. of inflection

6. **(10 pts)** Find two numbers x, y such that $x + y = 10$ and the quantity $x^2 + 4y^2$ is minimized.

$$y = 10 - x$$

$$\therefore \quad x^2 + 4y^2 = x^2 + 4(10-x)^2$$
$$= x^2 + 4(100 + x^2 - 20x)$$
$$= 5x^2 - 80x + 400$$

$f'(x) = 10x - 80$ $\Rightarrow$ Critical point $x = \frac{80}{10} = 8.$

$$\Rightarrow \quad y = 10 - x = 2.$$

So $\boxed{x = 8, \quad y = 2}$

7. **(10 pts)** Find the point (x, y) on the graph of $y = 1 + \sqrt{x}$ such that its distance from $(9/2, 1)$ is minimzed.

Distance $= \sqrt{\left(x - \frac{9}{2}\right)^2 + (y-1)^2}$

Since $y = 1 + \sqrt{x}$, we get:

$$f(x) = \sqrt{\left(x - \frac{9}{2}\right)^2 + (\sqrt{x})^2} = \sqrt{\left(x - \frac{9}{2}\right)^2 + x}$$

$$\therefore \quad f'(x) = \frac{1}{2}\left(\left(x - \frac{9}{2}\right)^2 + x\right)^{-1/2}\left(2\left(x - \frac{9}{2}\right) + 1\right)$$

$$= \frac{1}{2} \frac{(2x - 8)}{\sqrt{(x - 9/2)^2 + x}}$$

Denominator can never be 0 $\Rightarrow$ only critical point: $2x - 8 = 0$ $\Rightarrow x = \frac{8}{2} = 4$

So $\boxed{x = 4, \quad y = 1 + \sqrt{4} = 3}$

8. **(10 pts)** A rectangular box with a square base is to be constructed from two materials. The material for the square top and bottom costs \$ $3/ft^2$, and the material for the four rectangular sides costs \$ $2/ft^2$. Find the dimensions of the bos with largest volume which can be made for exactly \$ 72. Clearly indicate the length, width, height and volume of the box.

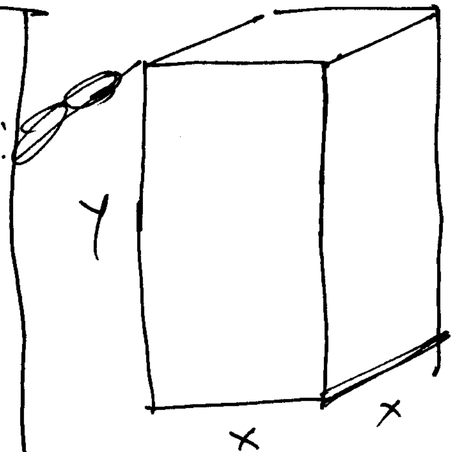
Final answer

Dimensions:

2, 2,

$y = \dfrac{72 - 6(2)^2}{8}$

$= 6$



$f'(x) = \dfrac{72-6x^2}{8} + x\left(\dfrac{-12x}{8}\right)$

$= \dfrac{72-6x^2}{8} - \dfrac{12x^2}{8}$

Setting to 0:

$\dfrac{72-18x^2}{8} = 0 \Rightarrow x = \sqrt{\dfrac{72}{18}} = 2$

Cost of making box = cost of top + bottom + cost of 4 sides.

Cost of top + bottom $= (2x^2) \times (3)$

total area in ft^2 — cost per ft^2

Cost of 4 sides $= (4xy) \times 2$

total area — cost per ft^2

Total cost $= 6x^2 + 8xy = 72. \Rightarrow y = \dfrac{72-6x^2}{8x}$

Maximize volume $= x^2 y$

$f(x) = x^{\cancel{2}}\left(\dfrac{72-6x^2}{8\cancel{x}}\right) = x\left(\dfrac{72-6x^2}{8}\right)$

9. **(8 pts)** Find the equation of the tangent line at $(-\frac{\pi^2}{1+\pi}, \pi)$ on the graph of

$$x\sec(y) = y^2 + xy.$$

Use implicit differentiation: $\frac{d}{dx}\left[x\sec(y) = y^2 + xy\right]$

$$\Rightarrow \sec(y) + x\sec(y)\tan(y)\frac{dy}{dx} = 2y\frac{dy}{dx} + y + x\frac{dy}{dx}$$

$$\Rightarrow \sec(y) - y = \left[2y + x - x\sec(y)\tan(y)\right]\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sec(y) - y}{2y + x - x\sec(y)\tan(y)}$$

$$\frac{dy}{dx} = \frac{\sec(\pi) - \pi}{2\pi + \left(\frac{-\pi^2}{1+\pi}\right) - \left(\frac{-\pi^2}{1+\pi}\right)\sec(\pi)\tan(\pi)}$$

So equation is:

$$y - \pi = \left(\frac{\sec(\pi) - \pi}{2\pi + \left(\frac{-\pi^2}{1+\pi}\right) + \frac{\pi^2}{1+\pi}\sec(\pi)\tan(\pi)}\right)\left(x + \frac{\pi^2}{1+\pi}\right)$$

10. **(7 pts)** The circumference C of a circle is increasing at the rate $\frac{dC}{dt} = 2ft/\ sec$. What is the rate of change $\frac{dA}{dt}$ of the area A of the circle when the radius of the circle is 6 ft ? The circumference C is given by $C = 2\pi r$ and the area is given by $A = \pi r^2$ where r is the radius of the circle.

$$\frac{d}{dt}\left[A = \pi r^2\right] \Rightarrow \frac{dA}{dt} = \pi 2r\frac{dr}{dt}$$

We need $\frac{dr}{dt}$:

$$\frac{d}{dt}\left[C = 2\pi r\right] \Rightarrow \frac{dC}{dt} = 2\pi\frac{dr}{dt}$$

So $2 = 2\pi\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{1}{\pi}$

$$\therefore \frac{dA}{dt} = \pi(2)(6)\left(\frac{1}{\pi}\right) = \boxed{12}$$

11. **(7 pts)** A point (x, y) is moving along a curve such that the rate of change of its x coordinate is given by $\frac{dx}{dt} = 3$ and the rate of change of its y coordinate is given by $\frac{dy}{dt} = -2$. Consider the line through this moving point and $(2, 5)$. Find the rate of change of the slope s of this line, when the moving point has coordinates $(1, 1)$.

Slope $s = \dfrac{y-5}{x-2}$

$$\frac{ds}{dt} = \frac{d}{dt}\left[\frac{y-5}{x-2}\right] = \frac{\frac{dy}{dt}(x-2) - (y-5)\frac{dx}{dt}}{(x-2)^2}$$

Plug in $x=1,\ y=1,\ \frac{dx}{dt} = 3,\ \frac{dy}{dt} = -2$

$$= \frac{(-2)(1-2) - (1-5)(3)}{(1-2)^2} = \frac{2+12}{(-1)^2} = 14$$

$$\boxed{\frac{ds}{dt} = 14}$$

12. **(8 pts)** Use the definition of the derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

to find the derivative of $f(x) = \sqrt{2x-1}$. DO NOT use the rules of derivatives to find the answer. Use limits.

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{2(x+h)-1} - \sqrt{2x-1}}{h}$$

Rationalizing the numerator:

$$\lim_{h \to 0} \frac{\sqrt{2(x+h)-1} - \sqrt{2x-1}}{h} \cdot \frac{\left(\sqrt{2(x+h)-1} + \sqrt{2x-1}\right)}{\sqrt{2(x+h)-1} + \sqrt{2x-1}}$$

$$= \lim_{h \to 0} \frac{(2(x+h)-1) - (2x-1)}{h\left(\sqrt{2(x+h)-1} + \sqrt{2x-1}\right)} = \lim_{h \to 0} \frac{2\cancel{h}}{\cancel{h}\left(\sqrt{2(x+h)-1} + \sqrt{2x-1}\right)}$$

$$= \frac{1}{\sqrt{2x-1} + \sqrt{2x-1}} = \boxed{\frac{1}{2\sqrt{2x-1}}}$$

13. **(5 pts)** Consider the following function :

$$f(x) = \begin{cases} x-2 & x<3 \\ 5 & x=3 \\ -x^2+10 & x>3 \end{cases}$$

Does $\lim_{x\to 3} f(x)$ exist ? If the limit does exist, write the value of this limit. Is the function continuous at $x = 3$? Show all your work and explain all your answers clearly.

$$\lim_{x\to 3^-} f(x) = 3-2 = 1$$

$$\lim_{x\to 3^+} f(x) = -(3)^2+10 = 1$$

Since left limit = right limit, $\lim_{x\to 3} f(x)$ exists and is equal to 1.

However, $f(3) = 5$ which is not equal to $\lim_{x\to 3} f(x) = 1$.

$\therefore$ $f(x)$ is not continuous at $\underline{\underline{x=3}}$