1. **(10 pts)** Find f_x, f_y and f_{xy} when $f(x, y) = \sqrt{x^4 + y^3}$.

$$f_x = \frac{1}{2\sqrt{x^4+y^3}} \cdot 4x^3$$

$$f_y = \frac{1}{2\sqrt{x^4+y^3}} \cdot 3y^2$$

$$f_{xy} = 2x^3 \left(-\frac{1}{2}\right) \frac{1}{(x^4+y^3)^{3/2}} \cdot 3y^2$$

$$= \frac{-3x^3y^2}{(x^4+y^3)^{3/2}}$$

2. Consider the function $f(x, y) = 2y^3 - 3x^2 - 3xy + 9x$.

(a) **(12 pts)** Find all critical points of $f(x, y)$. (Hint : There are two)

$$f_y = 6y^2 - 3x$$

$$f_x = -6x - 3y + 9$$

No undefined critical pts. So set both f_x & f_y to 0.

$$6y^2 - 3x = 0$$

$$-6x - 3y + 9 = 0$$

From first eq., $6y^2 = 3x \Rightarrow x = 2y^2$

substituting into second eq.

$$-6(2y^2) - 3y + 9 = 0 \Rightarrow -12y^2 - 3y + 9 = 0$$

$$\Rightarrow (-12y + 9)(y + 1) = 0$$

$$\Rightarrow y = -1 \quad \text{OR} \quad y = 9/12 = 3/4$$

Since $x = 2y^2$, critical pts :

$\boxed{y = -1, \; x = 2}$ AND $\boxed{y = 3/4, \; x = 9/8}$

(b) **(8 pts)** Decide whether each critical point found in (a) is a relative minimum, relative maximum, saddle point or indeterminable.

Apply Second Partial Derivative test :

$f_{xx} = -6$

$f_{yy} = 12y$

$f_{xy} = -3$

compute $d = f_{xx} f_{yy} - (f_{xy})^2$ at critical pts.

At $\boxed{y=-1 \ , x=2}$

$d = (-6)(-12) - (-3)^2$

$= 63$

$d > 0 \ , \ f_{xx} < 0 \implies$ relative maximum

At $\boxed{y = 3/4 \ , x = 9/8}$

$d = (-6)(9) - (-3)^2$

$= -63 < 0$

$\implies$ saddle point.

3. **(15 pts)** The volume of a cylinder is given by the formula $V = \pi r^2 h$ where r is the radius, and h is the height. Find the dimensions of the cylinder with largest volume subject to the constraint that the sum of twice the radius plus the height equals 9 inches.

maximize $\pi r^2 h$

subject to

$2r + h = 9$

Lagrange Multipliers :

$$F(r, h, \lambda) = \pi r^2 h - \lambda(2r + h - 9)$$

Critical pts of $F(r, h, \lambda)$:

$$\frac{\partial F}{\partial r} = 2\pi r h - 2\lambda$$

$$\frac{\partial F}{\partial h} = \pi r^2 - \lambda$$

$$\frac{\partial F}{\partial \lambda} = -2r - h + 9$$

setting all three to zero and eliminating λ

from first 2 eqns.

$2\pi r h = 2\lambda$ & $\pi r^2 = \lambda$ $\Rightarrow 2\pi r h = 2\pi r^2$

$\Rightarrow 2\pi r h - 2\pi r^2 = 0$

$\Rightarrow 2\pi r(h - r) = 0$

$\Rightarrow r = 0$ OR $h = r$

$r = 0$ means volume $= 0$, so no good. So $\boxed{h = r}$

Plugging into last eq. : $-2r - r + 9 = 0 \Rightarrow \boxed{r = 3 = h}$

4. **(10 pts)** Compute $\iint_R (x^2+y^2)\,dA$ where R is the square region with vertices $(0,0), (2,0), (2,2), (0,2)$.

$$\iint_R (x^2+y^2)\,dA = \int_0^2\int_0^2 (x^2+y^2)\,dx\,dy$$

$$= \int_0^2 \left[\frac{x^3}{3} + y^2 x\right]\Big|_0^2 dy$$

$$= \int_0^2 \left(\frac{8}{3} + 2y^2\right) dy$$

$$= \frac{8y}{3} + \frac{2y^3}{3}\Big]\Big|_0^2.$$

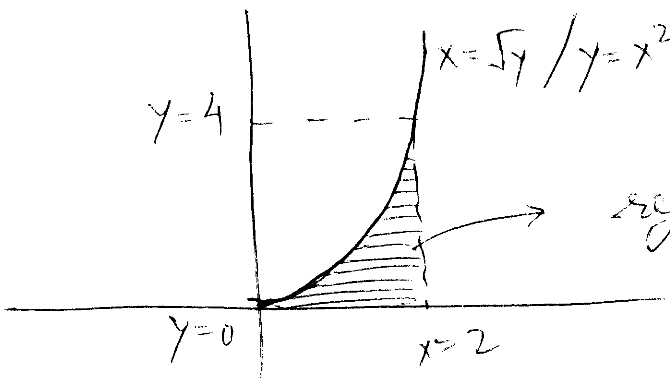
$$= \frac{16}{3} + \frac{16}{3}$$

$$= \boxed{\frac{32}{3}}$$

5. **(15 pts)** Compute $\int_0^4 \int_{y^{1/2}}^2 \frac{3}{\sqrt{1+x^3}} dx dy$. (Hint : You may consider changing the order of integration)

Change order of integration. First sketch region. From inner limits we get graphs

$x = 2$ + $x = \sqrt{y}$



$$\therefore \int_0^4 \int_{\sqrt{y}}^2 \frac{3}{\sqrt{1+x^3}} dx\, dy = \int_0^2 \int_{\cancel{0}\,0}^{x^2} \frac{3}{1+x^3} dy\, dx$$

$$= \int_0^2 \left(\frac{3y}{1+x^3} \Big|_0^{x^2} \right) dx$$

$$= \int_0^2 \frac{3x^2}{1+x^3} dx$$

$$= \ln|1+x^3| \Big|_0^2 = \ln(9) - \ln 1$$

$$= \boxed{\ln 9}$$

6. **(15 pts)** Minimize the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $2x - 2y + 6z = 44$

min $x^2 + y^2 + z^2$

subject to $2x - 2y + 6z = 44$

Use Lagrange Multipliers:

$$F(x, y, z, \lambda) = x^2 + y^2 + z^2 - \lambda(2x - 2y + 6z - 44)$$

$$\frac{\partial F}{\partial x} = 2x - 2\lambda \qquad \frac{\partial F}{\partial z} = 2z - 6\lambda$$

$$\frac{\partial F}{\partial y} = 2y + 2\lambda \qquad \frac{\partial F}{\partial \lambda} = (-2x + 2y - 6z + 44)$$

Setting to all to 0 we get:

$$2x = 2\lambda, \quad 2y + 2\lambda = 0 \quad + \quad 2z = 6\lambda$$

$$\Rightarrow \quad x = \lambda, \quad y = -\lambda \quad + \quad z = 3\lambda.$$

Substituting in last eq.

$$-2\lambda - 2\lambda - 18\lambda + 44 = 0$$

$$\Rightarrow \quad -22\lambda + 44 = 0 \quad \Rightarrow \quad \boxed{\lambda = 2}$$

So $\boxed{x = 2, \ y = -2, \ z = 6}$

$$f(x, y, z) = 2^2 + (-2)^2 + 6^2 = \boxed{44}$$

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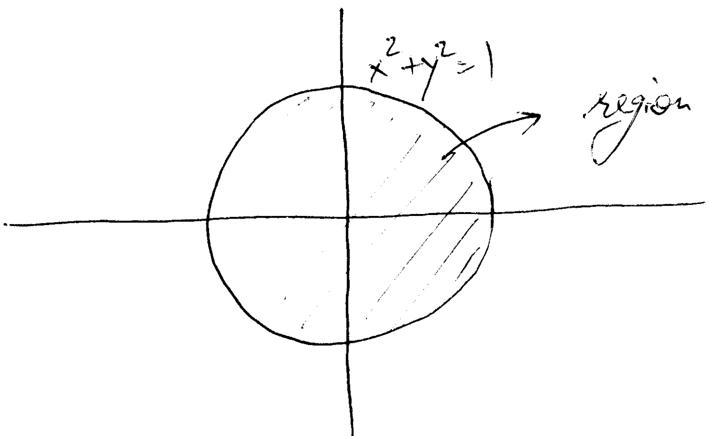
7. **(15 pts)** You will compute the volume of a hemisphere using double integrals. Consider the hemisphere whose base is the circle on the xy plane with the origin as the center and radius 1. So the equation of the base is given by :

$$x^2 + y^2 = 1$$

Set up (but DO NOT EVALUATE) the double integral to compute the volume of the hemisphere. (Hint : You have to figure out four things : a) which function of two variables are you going to integrate ? b) what is the region over which you will compute your double integral ? It might be useful to sketch this region c) Choose an order of integration in the double integral d) Decide the limits on the double integral ?



The hemisphere is ~~the~~ a part of the sphere if the base on the xy plane is $x^2+y^2=1$, then the sphere is $x^2+y^2+z^2=1$, because the radius is 1 + the center is the origin

$$\Rightarrow \quad Z = +\sqrt{1-x^2-y^2}$$

We take positive sign since we look at the top half.

$$\Rightarrow \quad \text{Volume} = \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}} +\sqrt{1-x^2-y^2}\,dx\,dy$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}} +\sqrt{1-x^2-y^2}\,dy\,dx$$