1. Find the following limits.

(a) **(4 pts)** (Hint : Divide out common factors)

$$\lim_{x\to 1} \frac{x^2-1}{x^2+x-2}.$$

$$= \lim_{x\to 1} \frac{\cancel{(x-1)}(x+1)}{(x+2)\cancel{(x-1)}}$$

$$= \frac{1+1}{1+2} = \boxed{\frac{2}{3}}$$

(b) **(4 pts)** (Hint : Rationalize the numerator and divide out)

$$\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}.$$

$$= \lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} \frac{\sqrt{x}+2}{\sqrt{x}+2}$$

$$= \lim_{x\to 4} \frac{(\sqrt{x})^2-2^2}{(x-4)(\sqrt{x}+2)} = \lim_{x\to 4} \frac{\cancel{x-4}}{\cancel{(x-4)}(\sqrt{x}+2)}$$

$$= \frac{1}{\sqrt{4}+2} = \boxed{\frac{1}{4}}$$

(c) **(4 pts)**

$$\lim_{x \to \infty} \frac{x^4 - 2x}{9x^5 + x^2 - 6}.$$

$$= 0$$

because degree of numerator > degree of denominator.

2. Differentiate the following functions. DO NOT SIMPLIFY ANSWERS.

(a) **(7 pts)**

$$y = \sqrt{\frac{3 - 2x}{\tan(x)}}$$

$$\frac{dy}{dx} = \boxed{\frac{1}{2}\left(\frac{3-2x}{\tan(x)}\right)^{-1/2}\left(\frac{(-2)\tan(x) - (3-2x)(\sec^2(x))}{\tan^2(x)}\right)}$$

(b) **(6 pts)**

$$y = x(sin(x))^2$$

$$\frac{dy}{dx} = \boxed{\sin^2(x) + x\left(2\sin(x)\cos(x)\right)}$$

3. **(5 pts)** Solve the equation for all values of $\theta$ in the interval $[0, 2\pi]$ :

$$\sin(2\theta) + 2\sin(\theta) = 0.$$

(Hint: Use the double angle formula for $\sin(2\theta)$.

$\sin(2\theta) = 2\sin\theta\cos\theta$

$\therefore$ we get $2\sin\theta\cos\theta + 2\sin\theta = 0$

$\Rightarrow 2\sin\theta(\cos\theta + 1) = 0$

$\Rightarrow \sin\theta = 0$ OR $\cos\theta = -1$

$\Rightarrow \theta = 0, \pi, 2\pi$ OR $\theta = \pi$

So $\boxed{\theta = 0, \pi, 2\pi}$

if the students don't have $2\pi$, that is fine also

4. Consider the function

$$f(x) = 3x + \frac{12}{x}.$$

(This question has two parts - both parts refer to this function.)

(a) **(10 pts)** Determine all the intervals where the function is increasing and decreasing, and find the relative minimum and relative maximum of $f(x)$. You need to say at what values of $x$ these relative extrema occur, and identify whether it is a relative minimum or relative maximum. You DO NOT need to report the value of the function at these points.

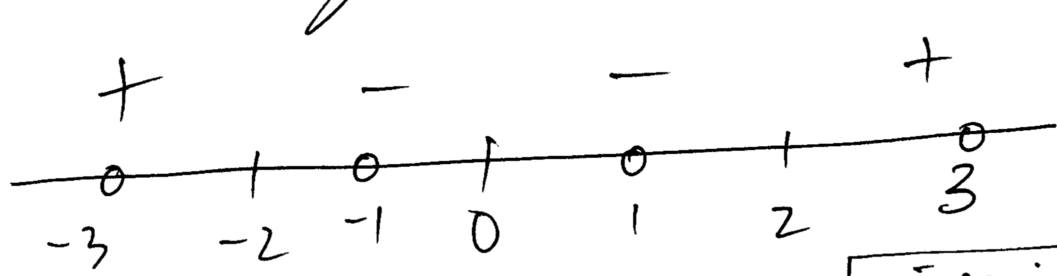
$$f'(x) = 3 - \frac{12}{x^2} = \frac{3x^2 - 12}{x^2}$$

Critical points : $f'(x) = \frac{3x^2-12}{x^2} = 0$ or undefined.

$$f'(x) = 0 \iff 3x^2 - 12 = 0$$

$$\Rightarrow \boxed{x = \pm 2.}$$

$f'(x)$ is undefined $\Rightarrow \boxed{x = 0.}$




$$f'(3) = f'(-3) = 3 - \frac{12}{(-3)^2} = 3 - \frac{12}{9} > 0$$

$$f'(1) = f'(-1) = 3 - \frac{12}{(-1)^2} = -9 < 0$$

increasing :
$(-\infty, -2), (2, \infty)$
decreasing :
$(-2, 0), (0, 2)$
Rel. min : $x = 2$
Rel. max : $x = -2$

(c) **(5 pts)** Find the absolute maximum and absolute minimum values of the function in the interval $1 \leq x \leq 3$.

Critical points in interval : $x = 2$.

$f(1) = 3(1) + \frac{12}{(1)} = 15$

$f(2) = 3(2) + \frac{12}{2} = 12$

$f(3) = 3(3) + \frac{12}{3} = 13$

Absolute min $= 12$

Absolute max $= 15$

5. **(10 pts)** Consider the function

$$x^3 - 5x^2 + 5x + 2.$$

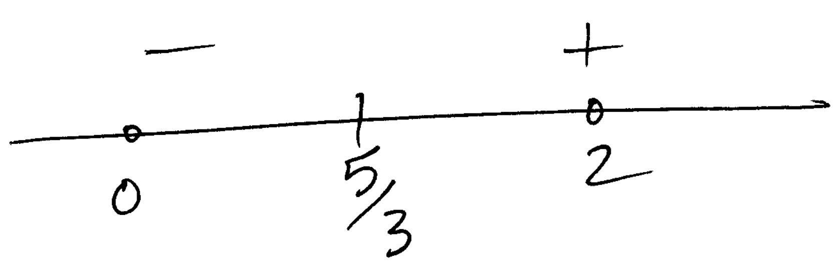
Determine the intervals where the function is concave upwards and concave downwards. Also indicate the points of inflection.

$f'(x) = 3x^2 - 10x + 5$

$f''(x) = 6x - 10$

$f''(x) = 0 \Rightarrow 6x - 10 = 0$

$\Rightarrow x = \frac{5}{3}$




$f''(0) = -10 < 0$

$f''(2) = 2 > 0$

Concave down : $(-\infty, 5/3)$

Concave up : $(5/3, \infty)$

Pt. of inflection : $\boxed{x = 5/3}$

6. **(10 pts)** Find two numbers $x, y$ such that $\frac{3}{2}x + y = 30$ and their product is maximized.

maximize $xy$. $y = 30 - \frac{3}{2}x$

$f(x) = x\left(30 - \frac{3}{2}x\right)$

$f'(x) = 30 - 3x$

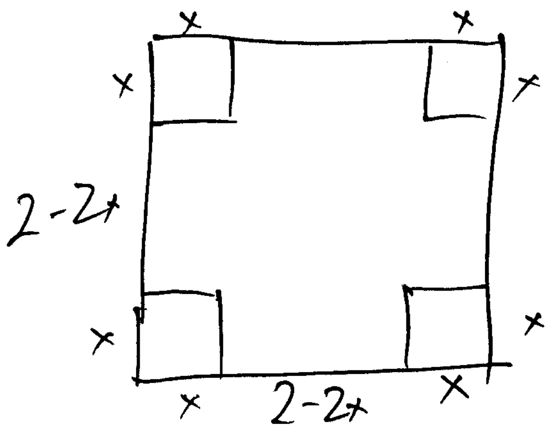
$f'(x) = 0 \Rightarrow 30 - 3x = 0$

$\boxed{x = 10}$

$y = 30 - \frac{3}{2}(10) = 15.$

$\boxed{x = 10 \text{ and } y = 15}$

7. **(15 pts)** An open box is to be made from a 2 ft by 2 ft rectangular piece of cardboard by cutting equal squares from the corners and turning up the sides. Find the largest volume of the box that can be made in this manner. Clearly indicate the length, width, height and volume of the box.




Volume $= x(2-2x)(2-2x) = x(4+4x^2-8x)$

$= 4x^3-8x^2+4x$

$f'(x) = 12x^2 - 16x + 4$

Setting to 0.

$4(3x^2-4x+1) = 0$

$(3x-1)(x-1) = 0$

$\Rightarrow \quad x = \frac{1}{3}, \ 1.$

At $x=1$, Volume $= 0.$

$x = \frac{1}{3}$, Volume $= \frac{1}{3}\left(\frac{4}{3}\right)\left(\frac{4}{3}\right) = \frac{16}{27}$

Therefore length $= \frac{4}{3}$, width $= \frac{4}{3}$, height $= \frac{1}{3}$.
Volume $= 16/27$

8. **(8 pts)** Find the equation of the tangent line at $(0, 2)$ on the graph of

$$\frac{d}{dx}\left[6x + y^2 = xy + 4.\right]$$

$$\Rightarrow \quad 6 + 2y\frac{dy}{dx} = y + x\frac{dy}{dx}$$

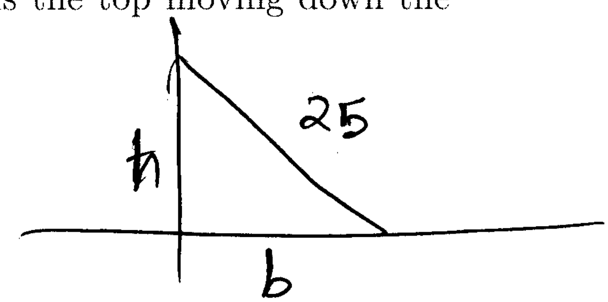
$$\Rightarrow \quad 6 - y = (x - 2y)\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{6-y}{x-2y} = \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{6-2}{0-2(2)} = -1$$

Equation : $y - 2 = (-1)(x - 0) \Rightarrow \boxed{x + y = 2}$

9. **(10 pts)** A 25 foot ladder is leaning against a house. The base of the ladder is pulled away at 2 ft/sec. How fast is the top moving down the wall when the bast is 7 ft from the house.




$$\frac{d}{dt}\left[h^2 + b^2 = 25^2\right]$$

$$2h\frac{dh}{dt} + 2b\frac{db}{dt} = 0.$$

When $b = 7$

$h = \sqrt{25^2 - 7^2}$

$$\frac{dh}{dt} = -\frac{b}{h}\frac{db}{dt} = \frac{-7(2)}{\sqrt{25^2 - 7^2}}$$

Speed $= \dfrac{14}{\sqrt{25^2 - 7^2}}$ (sign is not important)

10. **(7 pts)** Use the definition of the derivative $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ to find the derivative of $f(x) = \frac{1}{2x+1}$. DO NOT use the rules of derivatives to find the answer. Use limits.

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{2(x+h)+1} - \frac{1}{2x+1}}{h}$$

$$= \lim_{h \to 0} \frac{(2x+1) - (2x+2h+1)}{h(2x+1)(2x+2h+1)}$$

$$= \lim_{h \to 0} \frac{-2\cancel{h}}{\cancel{h}(2x+1)(2x+2h+1)}$$

$$\boxed{= \frac{-2}{(2x+1)^2}}$$

11. **(5 pts)** Consider the following function :

$$f(x) = \begin{cases} x-2 & x<3 \\ 5 & x=3 \\ -x^2+10 & x>3 \end{cases}$$

Does $\lim_{x\to 3} f(x)$ exist ? If the limit does exist, write the value of this limit. Is the function continuous at $x = 3$ ? Show all your work and explain all your answers clearly.

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} x-2 = 3-2 = 1.$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} -x^2+10 = -(3)^2+10 = 1.$$

Since $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$, $\boxed{\text{limit exists}}$

$$\boxed{\lim_{x\to 3} f(x) = 1.}$$

However. $f(3) = 5 \neq \lim_{x\to 3} f(x).$

$\boxed{\therefore \text{, function is NOT continuous}}$

12. **(10 pts)** The volume $V$ of a cube is increasing at the rate $\frac{dV}{dt} = 3\ in.^3/sec.$ What is the rate of change $\frac{dS}{dt}$ of its surface area $S$ when each edge is 2 inches long ? The relation between edge length $l$ and volume $V$ is given by $V = l^3$, and between $l$ and surface area $S$ is given by $S = 6l^2$.

$$\frac{dV}{dt} = 3l^2 \frac{dl}{dt} \Rightarrow 3 = 3(2)^2 \frac{dl}{dt}$$

$$\Rightarrow \frac{dl}{dt} = \frac{1}{4}.$$

$$\frac{dS}{dt} = 12l \frac{dl}{dt}$$

$$= 12(2)\left(\frac{1}{4}\right) = \boxed{6 \text{ inches/sec.}}$$