1. Let $f(x, y) = cos(1 + xy^2)$.

(a) **(4 pts)** Compute f_y.

$$\frac{\partial}{\partial y} \cos(1 + xy^2) = \boxed{-\sin(1 + xy^2)(2xy)}$$

(b) **(6 pts)** Compute f_{yx}.

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} (\cos(1 + xy^2))\right) = \frac{\partial}{\partial x} \left((2xy)(-\sin(1 + xy^2))\right)$$

$$= 2y(-\sin(1 + xy^2)) + (2xy)(-\cos(1 + xy^2))$$

$$(y^2)$$

$$= \boxed{-2y \sin(1 + xy^2) - 2xy^3 \cos(1 + xy^2)}$$

2. Consider the function $f(x, y) = 5 + x^2 - x^2y - y^2 - \frac{1}{3}y^3$.

(a) **(8 pts)** Find all critical points of $f(x, y)$.

$$\frac{\partial}{\partial x} f(x,y) = 2x - 2xy = 0 \qquad (1)$$

$$\frac{\partial}{\partial y} f(x,y) = -x^2 - 2y - y^2 = 0 \qquad (2)$$

~~Setting $2x = 2xy$~~ From (1)

$$2x(1-y) = 0$$

$\Rightarrow$ either $x = 0$ OR $y = 1$.

Case 1 : $x = 0$.

In (2) we plug in $x = 0$.

$$0 - 2y - y^2 = 0 \Rightarrow -y(y+2) = 0$$

$$\Rightarrow y = 0 \text{ OR } y = -2$$

So we get 2 critical pts: $\boxed{x=0,\ y=0}$ AND $\boxed{x=0,\ y=-2}$

Case 2 : $y = 1$

In (2) we plug $y = 1$: $-x^2 - 2 - 1 = 0$

$$\Rightarrow x^2 = -3$$

This has no solutions

So no critical pts from this case.

Case 3 : $\boxed{x = 0 \text{ AND } y = 1}$ does not satisfy (2).

$\boxed{\text{So 2 critical pts.}}$

(b) **(7 pts)** Decide whether each critical point found in (a) is a relative minimum, relative maximum, saddle point or indeterminable.

$f_{xx} = 2 - 2y$

$f_{yy} = -2 - 2y$

$f_{xy} = -2x$

At $\underline{x = 0,\ y = 0}$

$d = f_{xx} f_{yy} - (f_{xy})^2$

$= 2(-2) - 0^2 = -4 < 0$

$\Rightarrow$ $\boxed{\text{saddle point}}$

At $x = 0,\ y = -2$

$d = f_{xx} f_{yy} - (f_{xy})^2$

$= 6 \cdot 2 - 0^2 = 12 > 0$

$f_{xx} = 6 > 0$

$\Rightarrow$ $\boxed{\text{relative minimum}}$

3. Write the n-th term in the following sequences :

(a) **(2 pts)** $-\frac{1}{2}, \frac{4}{3}, -\frac{9}{4}, \frac{16}{5}, \ldots$

$$a_n = \frac{(-1)^n n^2}{n+1} \qquad a_1 = -\frac{1}{2}, \quad a_2 = \frac{4}{3}$$

(b) **(2 pts)** $2, 2, \frac{8}{6}, \frac{16}{24}, \frac{32}{120}, \ldots$

$$a_n = \frac{2^n}{(n)!}$$

(recall $0! = 1$)

$a_1 = 2$ $a_2 = 2$

(c) **(2 pts)** $1, \frac{5}{8}, \frac{7}{15}, \frac{9}{24}, \frac{11}{35}, \frac{13}{48}, \ldots$

$$a_n = \frac{2n+1}{(n+1)^2 - 1}$$

$a_1 = 1$

$a_2 = \frac{5}{8}$

4. Determine if the following sequences converge or diverge. If the sequence converges, write the limit.

(a) **(3 pts)** $a_n = \frac{2^n+1}{3 \cdot 2^n}$

$$\lim_{n\to\infty} \frac{2^n+1}{(3)2^n} = \lim_{n\to\infty} \frac{\left(\frac{2^n+1}{2^n}\right)}{(3)\frac{2^n}{2^n}}$$

$$= \lim_{n\to\infty} \frac{\left(1 + \frac{1}{2^n}\right)}{3} = \boxed{\frac{1}{3}}$$

$\boxed{\text{converges}}$

(b) **(3 pts)** $a_n = (-1)^n(\frac{1}{n^2+3})$

$$\lim_{n\to\infty} (-1)^n \left(\frac{1}{n^2+3}\right) = \boxed{0}$$

$\boxed{\text{converges.}}$

(c) **(3 pts)** $a_n = (-1)^n(\frac{n+1}{n})$

Does not converge

because the even terms get close to 1 and the odd terms ~~converge to~~ approach -1 so the sequence does not approach ONE single number.

5. Determine if the following series converge or diverge. Clearly explain why.

(a) **(6 pts)** $\sum_{n=0}^{\infty} \frac{n}{500n+79}$

n-th term test.

$$\lim_{n\to\infty} \frac{n}{500n+79} = \frac{1}{500} \neq 0$$

$\Rightarrow$ series diverges

(b) **(6 pts)** $\sum_{n=1}^{\infty} (2n)!(\frac{2}{3})^n$

Ratio test: $\lim_{n\to\infty} \left| \frac{(2(n+1))!\left(\frac{2}{3}\right)^{n+1}}{(2n)!\left(\frac{2}{3}\right)^n} \right|$

$\boxed{\text{Since } (2n+2)! = (2n+2)(2n+1)(2n)!}$

$$= \lim_{n\to\infty} \left| (2n+2)(2n+1)\left(\frac{2}{3}\right) \right|$$

$= \infty > 1 \Rightarrow$ $\boxed{\text{does not converge}}$

(c) **(6 pts)** $\sum_{n=1}^{\infty} \frac{1}{n^2\sqrt{n}}$

P- series test

$$\sum_{n=1}^{\infty} \frac{1}{n^2\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$$

$p > 1$

series converges

6. **(12 pts)** Find the radius and interval of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{3^{-n}}{n+1}(x+1)^n.$$

Use Ratio test:

$$\lim_{n\to\infty} \left| \frac{\left(\frac{3^{-(n+1)}(x+1)^{n+1}}{n+2}\right)}{\frac{3^{-n}(x+1)^n}{n+1}} \right| = \lim_{n\to\infty} \left| \frac{3^{-1}(x+1)(n+1)}{n+2} \right|$$

$$= \lim_{n\to\infty} \left| \frac{1}{3}(x+1)\left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right) \right| = \left| \frac{1}{3}(x+1) \right|$$

set this < 1

$$\Rightarrow \quad -1 < \frac{x+1}{3} < 1 \quad \Rightarrow \quad -3 < x+1 < 3$$

$$\Rightarrow \quad -1-3 < x < -1+3$$

$$\boxed{\begin{array}{l} \Rightarrow \text{Radius of convergence} = 3 \\ \text{Interval} = (-4, 2) \end{array}}$$

7. **(10 pts)** Find the sum $\sum_{n=0}^{\infty} \frac{1}{3^n 2^{n-2}}$.

$$\sum_{n=0}^{\infty} \frac{1}{3^n 2^{n-2}} = \sum_{n=0}^{\infty} \frac{2^2}{3^n 2^n} = \sum_{n=0}^{\infty} \frac{4}{6^n}$$

Geometric Series :

$$\sum_{n=0}^{\infty} \frac{4}{6^n} = \frac{4}{1-1/6} = \boxed{\frac{24}{5}}$$

8. Evaluate the following double integrals. (Remember that sometimes it helps to change the order of integration).

(a) **(12 pts)** $\int_0^2 \int_0^{\sqrt{x}} y(x-y^2)^3 dydx$.

$$\int_0^2 \int_0^{\sqrt{x}} y(x-y^2)^3 dy\, dx$$

For inner integral x is constant.

Use substitution

$u = x - y^2$

$du = -2ydy$

$$\int_0^x y(x-y^2)dy$$

$$= -\frac{1}{2}\int_0 u\,du = -\frac{1}{2}\frac{u^2}{2} = -\frac{x-y^2}{4}$$

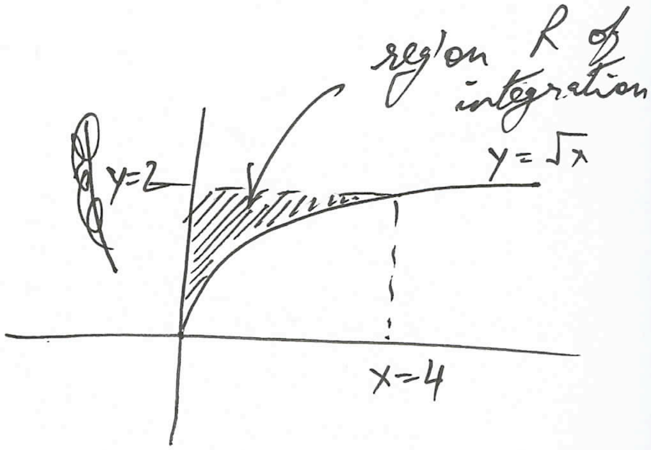
$$\therefore \int_0^{\sqrt{x}} y(x-y^2)\,dy = -\frac{x-y^2}{4}\Big|_0^{\sqrt{x}} = -\frac{x-(\sqrt{x})^2}{4} - \left(\frac{-x-0}{4}\right)$$

$$= -x/4$$

Outer integral

$$\int_0^2 \left(-\frac{x}{4}\right)dx = -\frac{x^2}{8}\Big|_0^2 = \boxed{-\frac{1}{2}}$$

(b) **(12 pts)** $\int_0^4 \int_{\sqrt{x}}^2 x sin(1+y^5) dydx$.



Change order of integration

$$\int_0^2 \int_0^{y^2} x \sin(1+y^5)\, dx\, dy$$

$$= \int_0^2 \left(\frac{x^2}{2} \sin(1+y^5) \Big|_0^{y^2} \right) dy$$

$$= \int_0^2 \frac{y^4}{2} \sin(1+y^5)\, dy$$

Now use substitution $u = 1+y^5$

$du = 5y^4 dy$

$$\Rightarrow \int \frac{y^4}{2} \sin(1+y^5) dy = \frac{1}{10} - \cos(1+y^5)$$

$$\Rightarrow \int_0^2 \frac{y^4}{2} \sin(1+y^5) dy = \frac{1}{10}\left(-\cos(1+y^5)\right)\Big|_0^2$$

$$= \boxed{\frac{-\cos(33) + \cos(1)}{10}}$$

9. Solve the following differential equations.

(a) **(10 pts)** $e^{x^2-x}y' + y = 2xy$.

$$e^{x^2-x}\frac{dy}{dx} = 2xy - y = (2x-1)y$$

separate variables :

$$\int \frac{1}{y}\,dy = \int \frac{(2x-1)}{e^{x^2-x}}\,dx$$

$$\ln|y| = \ln|e^{x^2-x}| + C$$

(b) **(10 pts)** $xy' - 2y = x^2 ln(x)$ with the initial conditions that $y = 0$ when $x = 1$.

$$y' - \frac{2y}{x} = \frac{x^2\ln(x)}{x}$$

$$P(x) = \frac{-2}{x} \qquad g(x) = x\ln(x)$$

$$\int P(x)\,dx = \int \frac{-2}{x}\,dx = -2\ln|x|$$

Integrating factor: $e^{-2\ln|x|} = x^{-2} = \frac{1}{x^2}$

$$\Rightarrow y = \frac{\int (x\ln(x))\cdot\frac{1}{x^2}\,dx}{1/x^2} = \frac{\left(\frac{(\ln x)^2}{2} + C\right)}{1/x^2}$$

$$= \boxed{\frac{x^2(\ln x)^2}{2} + Cx^2}$$

10. **(15 pts)** Approximate the definite integral $\int_0^1 xe^{-x^3}dx$ using a 7th-degree Taylor polynomial for the function xe^{-x^3}.

Power series for xe^{-x^3}

since $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$e^{-x^3} = 1 - \frac{x^3}{1!} + \frac{(-x^3)^2}{2!} - \frac{(x^3)^3}{3!} + \cdots$$

$$= 1 - \frac{x^3}{1!} + \frac{x^6}{2!} - \frac{x^9}{3!} + \cdots$$

so $$xe^{-x^3} = x\left(1 - \frac{x^3}{1!} + \frac{x^6}{2!} - \frac{x^9}{3!} + \cdots\right)$$

$$= x - \frac{x^4}{1!} + \frac{x^7}{2!} - \frac{x^{10}}{3!} + \cdots$$

7th degree : $x - x^4 + \frac{x^7}{2!}$

$$\int_0^1 xe^{-x^3}dx \approx \int_0^1\left(x - x^4 + \frac{x^7}{2}\right)dx = \frac{x^2}{2} - \frac{x^5}{5} + \frac{x^8}{16}\Bigg|_0^1$$

$$= \boxed{\frac{1}{2} - \frac{1}{5} + \frac{1}{16}}$$

11. **(11 pts)** Minimize the function $f(x,y,z) = x^2 + 2y^2 + 3z^2$ subject to the constraint $3x - 2y + z = \frac{34}{6}$. Find x, y, z at the minimum and the minimum value of the function.

$$F(x,y,z,\lambda) = x^2 + 2y^2 + 3z^2 - \lambda\left(3x - 2y + z - \frac{34}{6}\right)$$

$$F_x = 2x - 3\lambda = 0$$

$$F_y = 4y + 2\lambda = 0$$

$$F_z = 6z - \lambda = 0$$

$$F_\lambda = -3x + 2y - z + \frac{34}{6} = 0$$

From first 3 equations, we get.

$$x = \frac{3\lambda}{2}, \quad y = \frac{-\lambda}{2}, \quad z = \frac{\lambda}{6}$$

Substituting in last eq.

$$-3\left(\frac{3\lambda}{2}\right) + 2\left(-\frac{\lambda}{2}\right) - \left(\frac{\lambda}{6}\right) + \frac{34}{6} = 0$$

$$\Rightarrow -\frac{9\lambda}{2} - \lambda - \frac{\lambda}{6} = -\frac{34}{6}$$

$$\Rightarrow -27\lambda - 6\lambda - \lambda = -34$$

$$\Rightarrow \lambda = 1 \Rightarrow \boxed{x = \frac{3}{2}, \quad y = -\frac{1}{2}, \quad z = \frac{1}{6}}$$

$$f(x,y,z) = \left(\frac{3}{2}\right)^2 + 2\left(-\frac{1}{2}\right)^2 + 3\left(\frac{1}{6}\right)^2$$