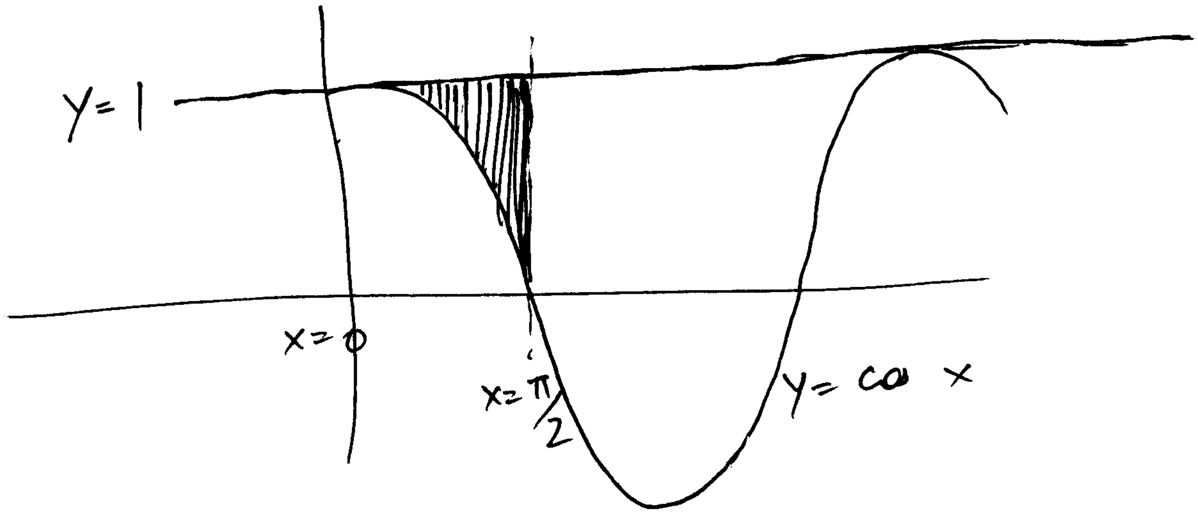
HW 4 Solutions MAT 17 B

4.

$y = \cos x$, $y = 1$, $x = 0$, $x = \pi/2$.

1. Sketch the curves :



2. $y=1$ is above $y=\cos x$.

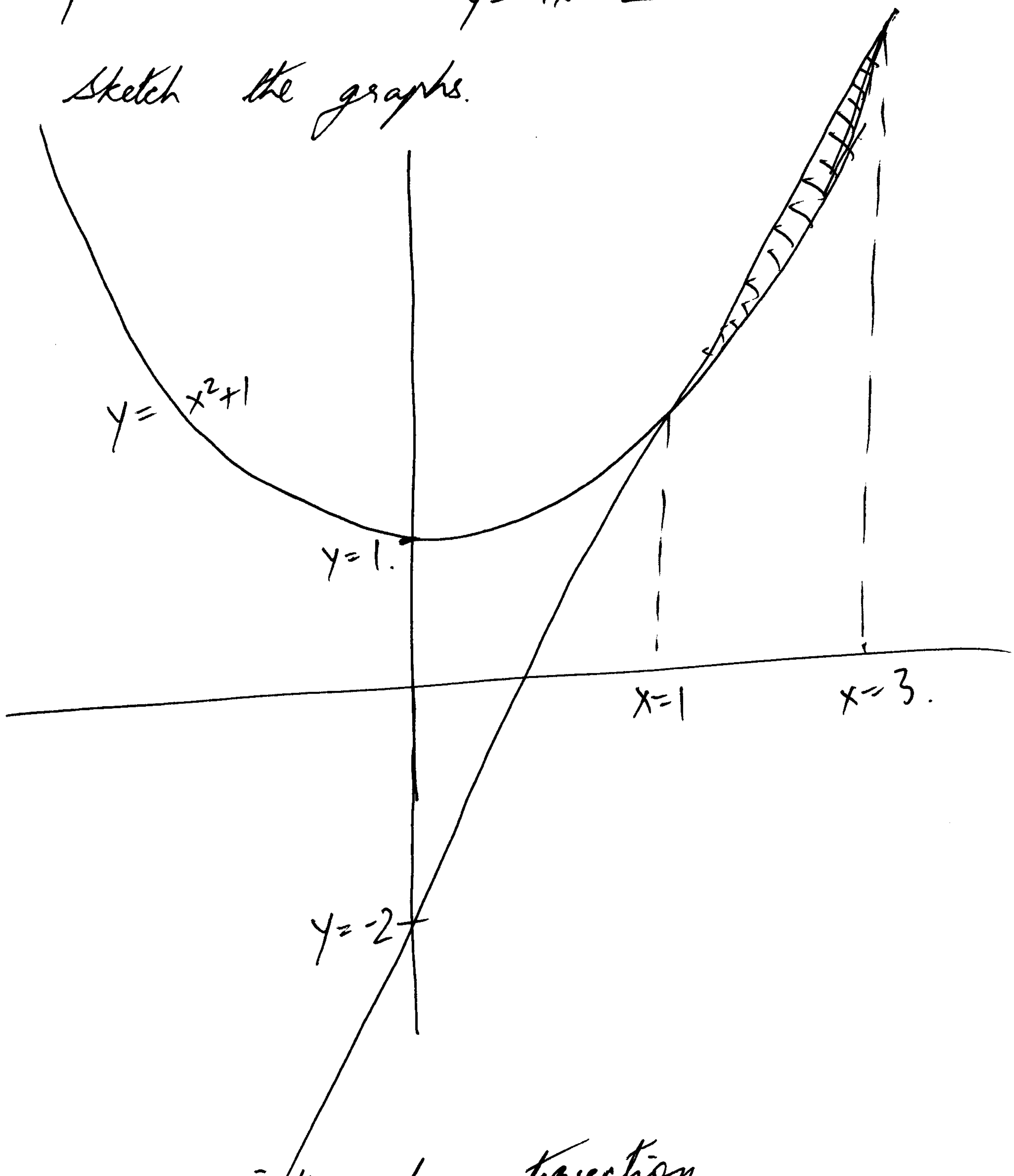
$$\therefore \text{ area } = \int_0^{\pi/2} (1 - \cos x)\, dx = x - \sin x \Big|_0^{\pi/2}$$

$$= \left(\frac{\pi}{2} - \sin\frac{\pi}{2}\right) - (0 - \sin 0)$$

$$= \boxed{\frac{\pi}{2} - \sin\frac{\pi}{2}}$$

5. $y = x^2+1$ $\qquad$ $y = 4x-2$

1. Sketch the graphs.



2. Find points of intersection

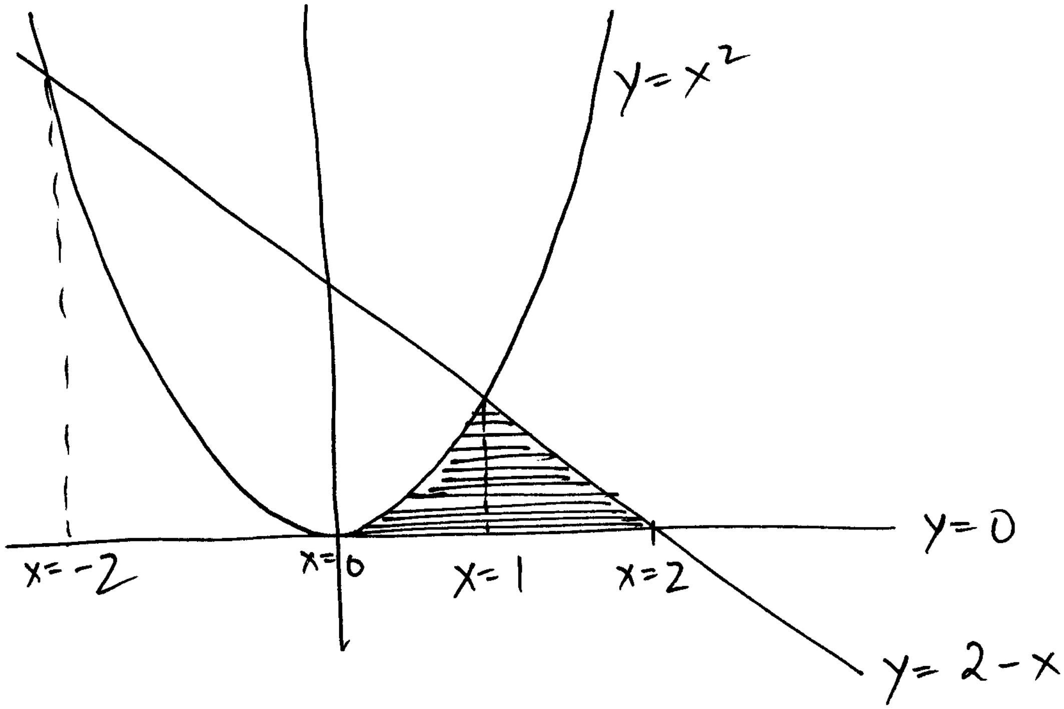
$$x^2+1 = 4x-2$$

$$\Rightarrow x^2-4x+3=0$$

$$\Rightarrow (x-3)(x-1)=0$$

$$\Rightarrow \boxed{x=3,\ x=1}$$

6. $y = x^2$, $y = 2 - x$, $y = 0$.



Need to split area into 2 parts, because top boundary changes from x^2 to $2-x$.

Need point of intersection of $y = x^2$ AND $y = 2 - x$.

$$\Rightarrow \quad x^2 = 2 - x$$

$$\Rightarrow \quad x^2 + x - 2 = 0$$

$$\Rightarrow \quad (x+2)(x-1) = 0$$

$$\Rightarrow \quad \boxed{x = -2, \; x = 1}$$

$x = 1$ is the relevant point.

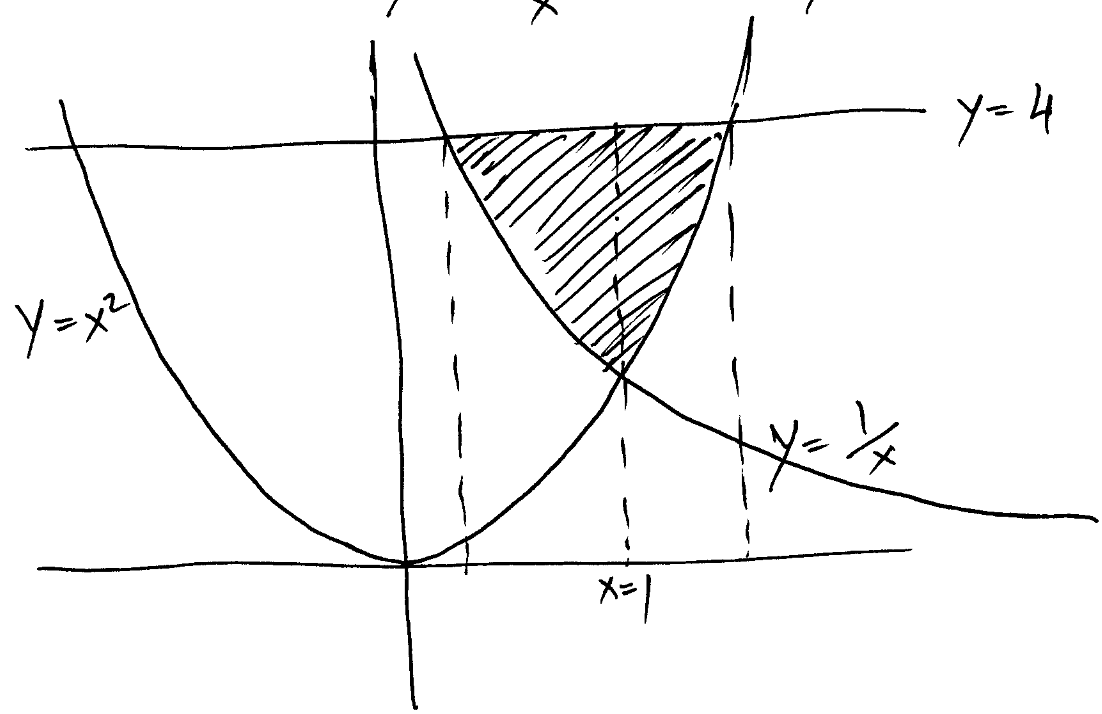
$$\text{Area} = \int_0^1 x^2 \, dx + \int_1^2 (2 - x) \, dx$$

$$= \left.\frac{x^3}{3}\right|_0^1 + \left.2x - \frac{x^2}{2}\right|_1^2$$

$$= \left(\frac{1}{3} - 0\right) + \left(\left(2(2) - \frac{2^2}{2}\right) - \left(2(1) - \frac{1^2}{2}\right)\right)$$

$$= \frac{1}{3} + 2 - 2 + \frac{1}{2} = \frac{5}{6}$$

7. $y = x^2$, $y = \frac{1}{x}$, $y = 4$.



Need to split area into 2 parts, because bottom boundary changes from $\frac{1}{x}$ to x^2.
Need to find the break pt as intersection point.

$$\frac{1}{x} = x^2 \Rightarrow x^3 = 1 \Rightarrow \boxed{x = 1}$$

For the first piece, we need left endpoint.

This is the intersection of $y = 4$ & $y = \frac{1}{x}$.

$\Rightarrow \quad 4 = \frac{1}{x} \quad \Rightarrow \quad \boxed{x = \frac{1}{4}}$

So first piece's area $= \displaystyle\int_{1/4}^{1} \left(4 - \frac{1}{x} \right) dx$

above

below.

For second piece, we need right endpt.

This is the intersection of $y = 4$ & $y = x^2$.

$\Rightarrow \quad 4 = x^2 \quad \Rightarrow \quad x = +2, -2.$

this is relevant for us.

So second piece's area

$$= \int_{1}^{2} (4 - x^2)\, dx$$

So total area $= \displaystyle\int_{1/4}^{1} \left(4 - \frac{1}{x}\right) dx + \int_{1}^{2} (4 - x^2)\, dx$

$$= 4x - \ln|x| \Big|_{1/4}^{1} + \left(4x - \frac{x^3}{3}\right)\Big|_{1}^{2}$$

$$= \left(4 - \ln|1|\right) - \left(1 - \ln \frac{1}{4}\right) + \left(8 - \frac{8}{3}\right) - \left(4 - \frac{1}{3}\right)$$

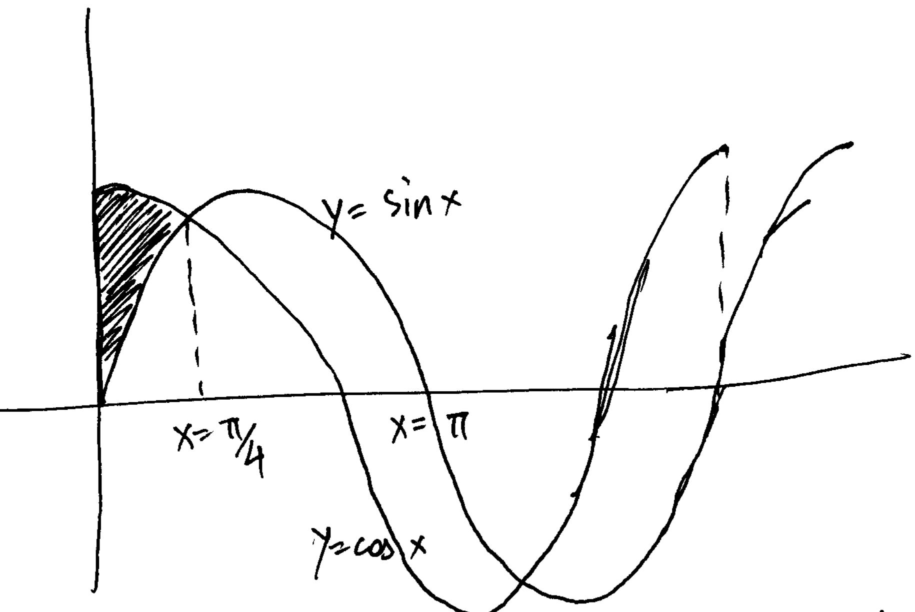
$$= \boxed{3 + \ln \frac{1}{4} + 4 - \frac{7}{3}}$$

8. $y = \sin x$

$y = \cos x$

between

$x = 0, \quad x = \pi/4$



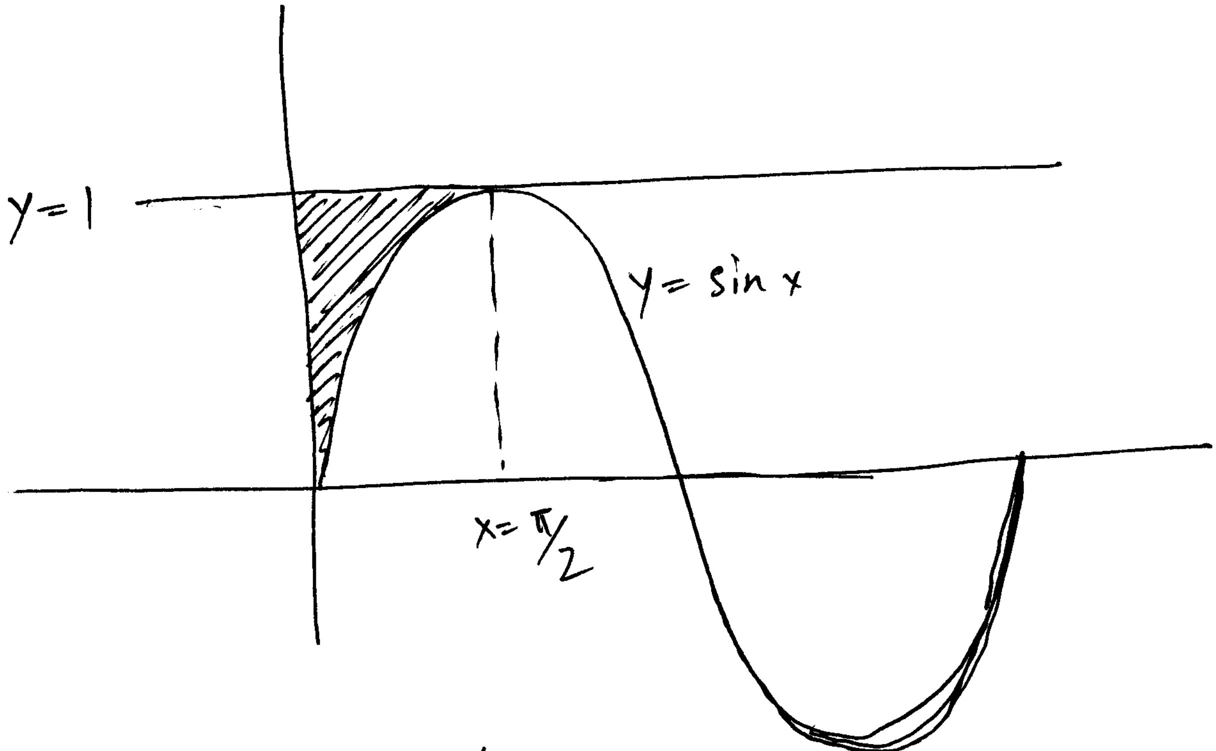
No need to split ⇒ both boundaries are a single function.

$$\text{Area} = \int_0^{\pi/4} \cos x - \sin x \, dx$$

$$= (\sin x + \cos x) \Big|_0^{\pi/4} = \left(\sin \frac{\pi}{4} + \cos \frac{\pi}{4}\right) - \left(\sin 0 + \cos 0\right)$$

$$= \boxed{\sin \frac{\pi}{4} + \cos \frac{\pi}{4} - 1}$$

9. $y = \sin x$, $y = 1$ from $x = 0$ to $x = \frac{\pi}{2}$



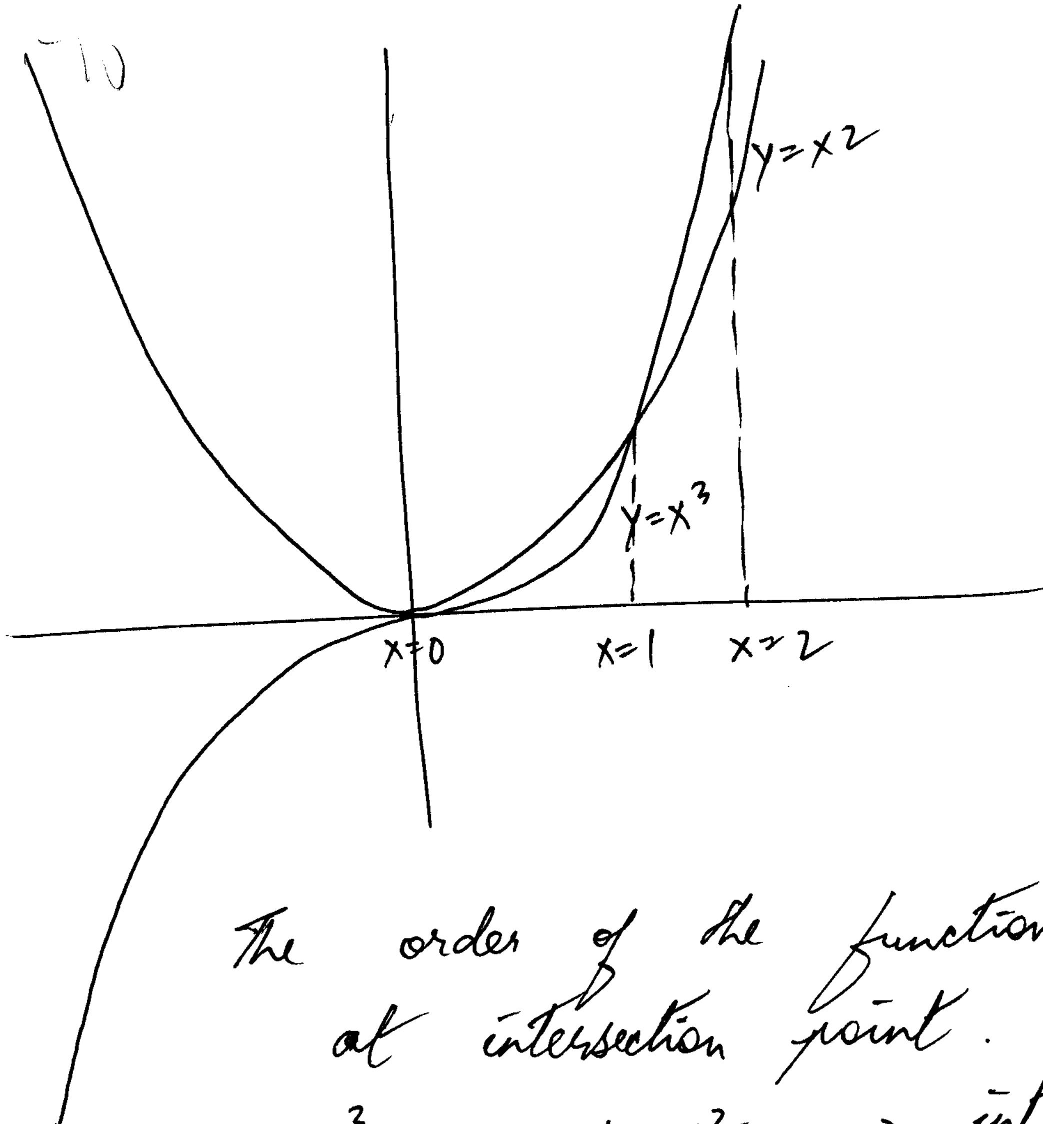
No need to split area
→ both boundaries are a single function.

$$\text{Area} = \int_0^{\pi/2} (1 - \sin x)\, dx = x + \cos x \Big|_0^{\pi/2}$$

$$= \left(\frac{\pi}{2} + \cos\frac{\pi}{2}\right) - (0 + \cos 0)$$

$$= \boxed{\frac{\pi}{2} + \cos\frac{\pi}{2} - 1}$$

11. $y=x^2$, $y=x^3$, $x=0$, $x=2$.



The order of the function changes at intersection point.

$y=x^3$, $y=x^2$ $\Rightarrow$ intersection point is $x^3=x^2$ $\Rightarrow$ $\boxed{x=1}$

Before $x=1$, $\underline{x^2}$ is above $\underline{x^3}$

After $x=1$, x^3 is above x^2

So. Area $= \displaystyle\int_0^1 (x^2-x^3)\,dx + \int_1^2 (x^3-x^2)\,dx$

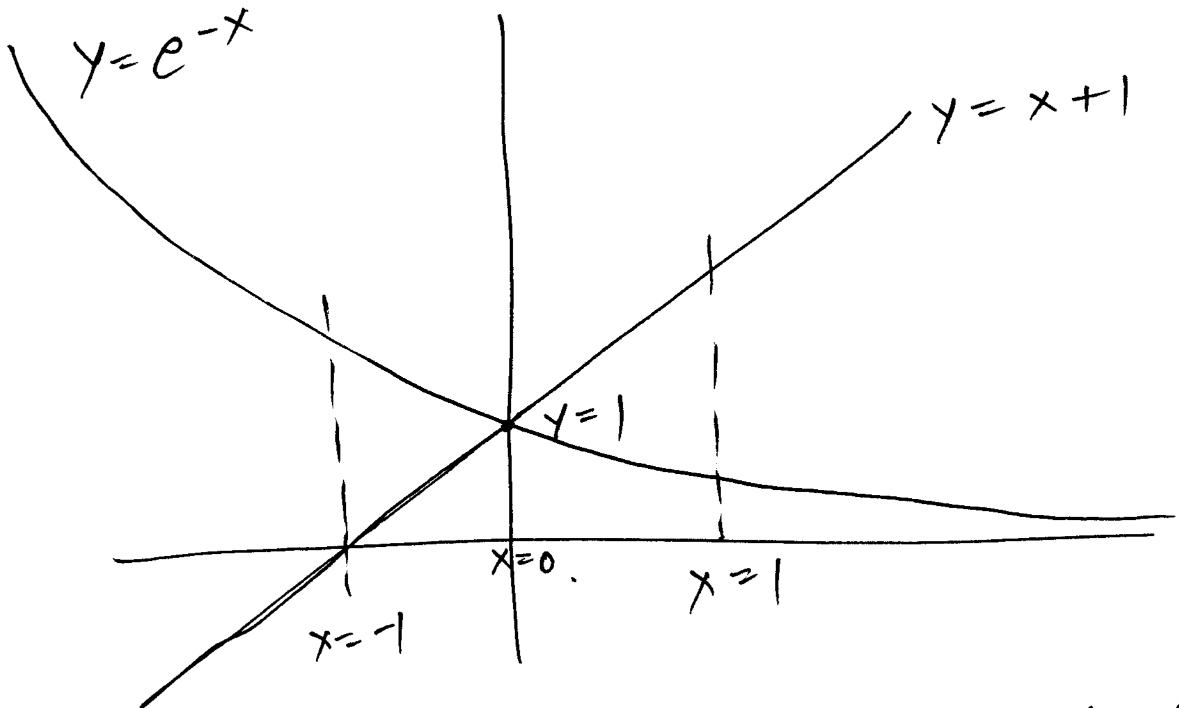
$$= \left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1 + \left(\frac{x^4}{4} - \frac{x^3}{3}\right)\Big|_1^2$$

$$= \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{2^4}{4} - \frac{2^3}{3}\right) - \left(\frac{1}{4} - \frac{1}{3}\right)$$

$$= \frac{1}{3} - \frac{1}{4} + 4 - \frac{8}{3} - \frac{1}{4} + \frac{1}{3}$$

$$= \boxed{1.5}$$

12. $y = e^{-x}$, $y = x+1$, $x = -1$, $x = 1$.



Order of functions changes about $x = 0$

Before $x=0$, $\underline{e^{-x}}$ is above $\underline{x+1}$

After $x=0$, $\underline{x+1}$ is above $\underline{e^{-x}}$.

So area $=$

$$\int_{-1}^{0} e^{-x} - (x+1)\, dx \quad + \quad \int_{0}^{1} x+1 - e^{-x}\, dx$$

$$= \left(-e^{-x} - \frac{x^2}{2} - x\right)\Big|_{-1}^{0} + \left(\frac{x^2}{2} + x + e^{-x}\right)\Big|_{0}^{1}$$

$$= \left(-e^{0} - 0 - 0\right) - \left(-e^{1} - \frac{(-1)^2}{2} - (-1)\right) + \left(\frac{1}{2} + 1 + e^{-1}\right) - \left(0 + e^{0}\right)$$

$$= -1 + e + \frac{1}{2} - 1 + \frac{1}{2} + 1 + e^{-1} - 1$$

$$= \boxed{e + e^{-1} - 1}$$