HW 17 Solutions MAT 17 B

1. $\frac{dS}{dt}$ = Rate in − Rate out.

Rate in: 2 gallons/minute enters with $\frac{1}{2}$ pound/gallon

$\therefore$ $2 \times \frac{1}{2} = 1$ pound enters per minute.

Rate out: 2 gallons of well-stirred solution leaves per minute.

How many pounds leave in 2 gallons?

$S(t)$ pounds in 100 gallons at time t

(gallons in tank does not change).

$\therefore$ $2\frac{S(t)}{100}$ pounds in 2 gallons.

$$\therefore \quad \frac{dS}{dt} = 1 - \frac{2S}{100} = 1 - \frac{S}{50}.$$

$$\Rightarrow \quad \frac{dS}{dt} + \frac{S}{50} = 1$$

Use FOLDE with $P(t) = \frac{1}{50}$, $Q(t) = 1$.

1. $\int P(t)\,dt = \int \frac{1}{50}\,dt = \frac{t}{50}$

2. $u(t) = e^{\int P(t)\,dt} = e^{t/50}$

3. $\int Q(t)u(t)\,dt = \int (1)\, e^{t/50}\, dt = 50e^{t/50}$

Final Solution : $\frac{1}{e^{t/50}}\left[50e^{t/50} + C\right]$.

$\therefore S(t) = 50 + Ce^{-t/50}$

At $t=0$, $S(0)=5$ (tank initially contains 5 pounds)

$\therefore 5 = 50 + Ce^{-0/50} \Rightarrow C = -45.$

$\therefore S(t) = 50 - 45e^{-t/50}.$

At $t=10$,

$S(10) = 50 - 45e^{-10/50} = 50 - 45e^{-1/5}$

At $t=60$ (1 hour)

$S(60) = 50 - 45e^{-60/50} = 50 - 45e^{-6/5}$

2. a) $200 - t$ gallons at time t. $\therefore$ $100-20 = 80$ gallons after 20 minutes

b). $\frac{dS}{dt}$ = Rate in − Rate out.

Rate in : Pure water comes in.

$\therefore$ Rate in $= 0$.

Rate out : 5 gallons of solution leaves per minute.

How many pounds of salt in 5 gallons?

At time t, $200-t$ gallons contain $S(t)$ pounds.

$\therefore$ 5 gallons would contain $\frac{5S(t)}{200-t}$ pound

$\therefore$ Rate out $= \frac{5S}{200-t}$ pounds per minute.

$$\therefore \frac{dS}{dt} = 0 - \frac{5S}{200-t} = \frac{-5S}{200-t}$$

$$\Rightarrow \int \frac{1}{S}\,dS = \int \frac{-5}{200-t}\,dt$$

$$\Rightarrow \ln|S| = +5\ln|200-t| + C$$

Exponentiating :

$$|S| = e^{5\ln|200-t|+C} = e^{C}e^{5\ln|200-t|}$$

$$S = \pm e^{c} e^{\ln|200-t|^5}$$

$$S = C_1 (200-t)^5$$

At $t=0$, $S(0) = 50$.

$$\therefore \quad 50 = C_1 (200)^5 \Rightarrow C_1 = 50 \times 200^{-5}$$

$\therefore$ After 20 minutes,

$$t=20, \quad S(20) = 50 \times 200^{-5} (200-20)^5$$

$$= 50 \times 200^{-5} \times 180^5$$

After 2 hours,

$$t = 120, \quad S(120) = 50 \times 200^{-5} (200-120)^5$$

$$= 50 \times 200^{-5} \times 80^5$$

c.) ~~200 gallons~~ $200 - t = 0$ for tank to be empty.

$$\Rightarrow \underline{t = 200 \text{ minutes}}$$

3. a). 6 gallons / min enters,

4 gallons / min leave,

$\therefore$ At time t, we have $\underline{100 + 2t \text{ gallons}}$.

$\therefore$ At $\underline{t = 10}$, we have $\underline{120 \text{ gallons}}$.

After 1 hour, $t = 60$, we have $\underline{220 \text{ gallons}}$.

b). $$\frac{dS}{dt} = \text{Rate in} - \text{Rate out}.$$

<u>Rate in</u>: New Solution enters @ 6 gallons / minute containing $\frac{1}{2}$ pound / gallon.

$\therefore$ $6 \times \frac{1}{2} = 3$ pounds / minute enters.

<u>Rate out</u>: Well-stirred solution in tank drained @ 4 gallons / minute. How many pounds in 4 gallons?

At time t, $S(t)$ pounds in $100 + 2t$ gallons.

$\therefore$ $\dfrac{4\, S(t)}{100 + 2t}$ pounds in 4 gallons.

$$\therefore \quad \frac{dS}{dt} = 3 - \frac{4S}{100 + 2t}$$

$$\Rightarrow \quad \frac{dS}{dt} + \frac{4S}{100+2t} = 3.$$

Use FOLDE with $P(t) = \dfrac{4}{100+2t}$, $Q(t) = 3$.

1. $\displaystyle\int P(t)\,dt = \int \frac{4}{100+2t}\,dt = 2\ln|100+2t|$

2. $u(t) = e^{\int P(t)\,dt} = e^{2\ln|100+2t|} = (100+2t)^2$

3. $\displaystyle\int Q(t)\,u(t)\,dt = \int 3(100+2t)^2\,dt$

$$= \frac{3}{2}\,\frac{(100+2t)^3}{3} = \frac{(100+2t)^3}{2}$$

Final Solution:

$$S(t) = \frac{1}{(100+2t)^2}\left[\frac{(100+2t)^3}{2} + C\right]$$

$$= \frac{100+2t}{2} + C(100+2t)^{-2}$$

At $t = 0$, $S(0) = 10$

$$\therefore \quad 10 = 50 + C(100)^{-2}$$

$$\Rightarrow C = -40 \times 100^2$$

$$\therefore \quad S(t) = (50+t) + (-40\times 100^2)(100+2t)^{-2}$$

At $t = 10$, $S(10) = 60 + (-40 \times 100^2)(120)^{-2}$

$$= 60 - 40 \times \left(\frac{10}{12}\right)^2$$

At $t = 60$ (1 hour)

$$S(60) = 110 + (-40 \times 100^2)(220)^{-2}$$

$$= 110 - 40\left(\frac{10}{22}\right)^2$$

4 a). 3 gallons enter per minute.

4 gallons leave per minute.

$\therefore$ After t minutes,

$400 - t$ gallons are left.

After 10 minutes we have $400 - 10 = \underline{390 \text{ gallons}}$.

After 60 minutes we have $400 - 60 = \underline{340 \text{ gallons}}$.

After 6 hours & 40 minutes, $t = 400$

we have $400 - 400 = \underline{0 \text{ gallons}}$.

b) Let $S(t)$ = amount in gallons of alcohol.

$S(0) = 3\%$ of $400 = 12$ gallons.

$\frac{dS}{dt}$ = Rate in − Rate out.

Rate in : @ 3 gallons / minute with 6% alcohol.

$= \frac{6}{100} \times 3$ gallons / minute.

Rate out : 4 gallons / minute leave.

At time t, $S(t)$ gallons of alchol in $400-t$ gallons.

$\therefore$ $4\frac{S(t)}{400-t}$ in 4 gallons.

$$\therefore \frac{dS}{dt} = \frac{18}{100} - \frac{4S}{400-t}.$$

$$\Rightarrow \frac{dS}{dt} + \frac{4S}{400-t} = \frac{18}{100}$$

$$P(t) = \frac{4}{400-t}, \quad Q(t) = \frac{18}{100}$$

1. $\int P(t)dt = \int \frac{4}{400-t}dt$

$= -4\ln|400-t|$

2. $u(t) = e^{\int P(t)dt} = e^{-4\ln|400-t|} = e^{\ln|400-t|^{-4}}$

$= (400-t)^{-4}$

3. $\int Q(t)u(t)dt = \int \frac{18}{100}(400-t)^{-4}dt$

$= \frac{18}{100}\int (400-t)^{-4}dt$

Substitute $u = 400-t$

$du = -dt$

$\therefore$

$\frac{18}{100}\int (400-t)^{-4}dt = \frac{18}{100}\int u^{-4}(-du)$

$= -\frac{18}{100}\frac{u^{-3}}{-3}$

$= \frac{6}{100}u^{-3}$

$= \frac{6}{100}(400-t)^{-3}$

Final solution :

$$\frac{1}{(400-t)^{-4}}\left[\frac{6}{100}(400-t)^{-3}+C\right]$$

$$= \frac{6}{100}(400-t) + \frac{C}{(400-t)^{-4}}$$

At $t=0$, $S(0) = 12$.

$$\therefore \quad 12 = \frac{6}{100}\times 400 + \frac{C}{(400)^{-4}}$$

$$12 = 24 + C(400)^4$$

$$\Rightarrow \quad C = -\frac{12}{(400)^4}$$

$$\therefore \quad \boxed{S(t) = \frac{6}{100}(400-t) + \frac{-12}{(400)^4}\times(400-t)^4}$$

At $t=10$, $S(10) = \frac{6}{100}(400-10) - \frac{12}{400^4}(400-10)^4$

$$= \boxed{\frac{6}{100}(390) - \frac{12}{400^4}(390)^4}$$

At $t = 60$ (1 hour)

$$S(60) = \frac{6}{100}(400-60) + \frac{-12}{400^4}(400-60)^4$$

$$= \boxed{\frac{6}{100} \times 340 - 12\left(\frac{340}{400}\right)^4}$$