HW 14 Solutions MAT 17B

1. $\frac{dy}{dx} = y(2-y)$

Equilibrium points : Set $y(2-y) = 0$

$\Rightarrow y = 0$ or $y = 2$.

Stability : $g(y) = y(2-y) = 2y - y^2$

$g'(y) = 2 - 2y$

At $y = 0$, $g'(y) = 2 - 2(0) = 2 > 0$

$\therefore$ Unstable equilibrium

At $y = 2$, $g'(y) = 2 - 2(2) = -2 < 0$

$\therefore$ Locally stable equilibrium

2. $\frac{dy}{dx} = (4-y)(5-y)$

Equilibrium points : Set $(4-y)(5-y) = 0$

$\Rightarrow y = 4, 5$.

Stability : $g(y) = (4-y)(5-y) = 20 - 9y + y^2$

$g'(y) = -9 + 2y$

At $y = 4$, $g'(y) = -9 + 2(4) = -1 < 0$

$\therefore$ Locally stable equilibrium

At $y = 5$, $g'(y) = -9 + 2(5) = 1 > 0$

$\therefore$ Unstable equilibrium.

3. $\frac{dy}{dx} = y(y-1)(y-2)$

Equilibrium points : $y(y-1)(y-2) = 0$

$\Rightarrow \quad y = 0, 1, 2.$

~~Stabth~~ Stability : $g(y) = y(y-1)(y-2) = y^3 - 3y^2 + 2y$

$g'(y) = 3y^2 - 6y + 2$

$\therefore$ At $y=0$, $g'(y) = 3(0)^2 - 6(0) + 2$

$= 2 > 0$

$\therefore$ Unstable equilibrium

At $y=1$, $g'(y) = 3(1)^2 - 6(1) + 2 = -1 < 0$

$\therefore$ Locally stable equilibrium

At $y=2$, $g'(y) = 3(2)^2 - 6(2) + 2 = 2 > 0$

$\therefore$ Unstable equilibrium

4. $\frac{dy}{dx} = y(2-y)(y-3)$

<u>Equilibrium points</u>: $y(2-y)(y-3) = 0$

$\Rightarrow y = 0, 2, 3.$

<u>Stability</u>: $g(y) = y(2-y)(y-3)$

$= -y^3 + 5y^2 - 6y$

$\therefore g'(y) = -3y^2 + 10y - 6$

<u>At $y=0$</u>, $g'(y) = -3(0)^2 + 10(0) - 6 = -6 < 0$

$\therefore$ <u>Locally stable equilibrium</u>

<u>At $y=2$</u>, $g'(y) = -3(2)^2 + 10(2) - 6$

$= 2 > 0$

$\therefore$ <u>Unstable equilibrium</u>

<u>At $y=3$</u>, $g'(y) = -3(3)^2 + 10(3) - 6$

$= -3 < 0$

$\therefore$ <u>Locally stable equilibrium</u>

5.

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

r = rate of growth = 1.5.

K = carrying capacity = 100

$\therefore$ we have $\displaystyle \frac{dN}{dt} = 1.5\,N\left(1 - \frac{N}{100}\right)$

Equilibrium points: set $\displaystyle 1.5N\left(1 - \frac{N}{100}\right) = 0$

$$\Rightarrow \quad 1.5N\left(\frac{100-N}{100}\right) = 0$$

$$\Rightarrow \quad N = 0 \quad \text{or} \quad \frac{100-N}{100} = 0$$

$$\Rightarrow \quad \underline{N = 0} \quad \text{or} \quad \underline{N = 100}$$

Stability: $\displaystyle g(N) = 1.5\,N\left(1 - \frac{N}{100}\right)$

$$= 1.5N - \frac{1.5N^2}{100}$$

$$\therefore \; g'(N) = 1.5 - \frac{1.5}{100}(2N) = 1.5 - \frac{3N}{100}$$

At $N = 0$,

$$g'(N) = 1.5 - \frac{3(0)}{100} = 1.5 > 0$$

$\therefore$ Unstable equilibrium

At $N = 100$,

$$g'(N) = 1.5 - \frac{3(100)}{100} = -1.5 < 0$$

$\therefore$ Locally stable equilibrium

6. $$\frac{dN}{dt} = N\left(1 - \frac{N}{50}\right) - \frac{9N}{5+N}$$

$$= N\left(\frac{50-N}{50}\right) - \frac{9N}{5+N}$$

$$= \frac{N(50-N)(5+N) - (9N)(50)}{50(5+N)}$$

$$= \frac{N(250 \cancel{} + 45N - N^2) - 450N}{50(5+N)}$$

$$= \frac{N(250 + 45N - N^2 - 450)}{50(5+N)}$$

$$= \frac{N(-200+45N-N^2)}{50(5+N)}$$

$$= \frac{-N(N^2-45N+200)}{50(5+N)}$$

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$$= \frac{-N(N-5)(N-40)}{50(5+N)}$$

Equilibrium points:

Set $\frac{-N(N-5)(N-40)}{50(5+N)} = 0$

$\Rightarrow \quad -N(N-5)(N-40) = 0$

$\Rightarrow \quad N = \underline{0,\ 5,\ 40}.$

Stability $g(N) = N\left(1-\frac{N}{50}\right) - \frac{9N}{5+N}$

$$= N - \frac{N^2}{50} - \frac{9N}{5+N}$$

$$\therefore g'(N) = 1 - \frac{2N}{50} - \left(\frac{9}{5+N} - \frac{9N}{(5+N)^2}\right)$$

[Use product Rule OR quotient rule for derivatives

$$= 1 - \frac{2N}{50} - \frac{9(5+N) - 9N}{(5+N)^2}$$

At $N=0$,

$$g'(N) = 1 - \frac{2(0)}{50} - \frac{9(5+0) - 9(0)}{(5+0)^2}$$

$$= 1 - \frac{36}{25} = \frac{-11}{25} < 0$$

$\therefore$ Locally stable equilibrium

At $N=5$,

$$g'(N) = 1 - \frac{2(5)}{50} - \frac{9(5+5) - 9(5)}{(5+5)^2}$$

$$= 1 - \frac{1}{5} - \frac{45}{100} = 1 - \frac{65}{100} = \frac{35}{100} > 0$$

$\therefore$ Unstable equilibrium

At $N=40$,

$$g'(N) = 1 - \frac{2(40)}{50} - \frac{9(5+40) - 9(40)}{(5+40)^2}$$

$$= 1 - \frac{8}{5} - \frac{45}{(45)^2} = 1 - \frac{8}{5} - \frac{1}{45}$$

$$= 1 - \frac{73}{45} = \frac{-28}{45} < 0$$

$\Rightarrow$ Locally stable equilibrium

10. $$\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right) - hN$$

and $r=2$, $K=1000$

$$\therefore \quad g(N) = 2N\left(1-\frac{N}{1000}\right) - hN$$

$$= 2N\left(\frac{1000-N}{1000}\right) - hN$$

$$= N\left(\frac{2(1000-N)}{1000} - h\right)$$

$$= N\left(\frac{2000-2N-1000h}{1000}\right)$$

$\therefore$ Equilibrium points :

$N=0$ OR

$$\frac{2000-2N-1000h}{1000} = 0$$

$$\Rightarrow \quad 2000-2N-1000h = 0$$

$$\Rightarrow \quad 2000-1000h = 2N$$

$$\Rightarrow \quad \frac{2000-1000h}{2} = N$$

$$\Rightarrow \quad 1000-500h = N$$

Stability : $g(N) = 2N\left(1 - \frac{N}{100}\right) - hN$

$$= 2N - \frac{2N^2}{100} - hN$$

$$\therefore \; g'(N) = 2 - \frac{N}{25} - h$$

At $N = 0$,

$$g'(N) = 2 - \frac{0}{25} - h = 2 - h$$

If $h > 2$, then $2 - h < 0$

$\Rightarrow$ locally stable equilibrium

If $h < 2$, then $2 - h > 0$

$\Rightarrow$ unstable equilibrium

Interesting Interpretation: If $h > 2$ then the population goes back to 0, since 0 is a stable equilibrium. If $h < 2$, then the population grows if perturbed a little from 0 because 0 is an unstable equilibrium

At $\underline{N = 1000 - 500h}$

$$g'(N) = 2 - \frac{1000 - 500h}{25} - h$$

$$= 2 - \frac{1000 - 500h - 25h}{25}$$

$$= 2 - \frac{1000 - 525h}{25}$$

$$= \frac{50 - 1000 - 525h}{25}$$

$$= \frac{-950 - 525h}{25}$$

Assuming $h \geq 0$, $g'(N)$ is < 0 for $N = 1000 - 500h$.

$\therefore$ <u>It is a locally stable equilibrium point</u>