HW 13 Solutions MAT 17 B

9. $\frac{dV}{dt} = 1 + \cos t$

Use separation of variables:

$$dV = (1 + \cos t)dt$$

$$\Rightarrow \int dV = \int (1 + \cos t)\, dt$$

$$\Rightarrow V = t + \sin t + C$$

We know $V(0) = 5$

$$\therefore 5 = 0 + \sin 0 + C$$

$$\Rightarrow \boxed{C = 5}$$

So $\boxed{V(t) = t + \sin t + 5}$

NOTE: Answer at the end of the textbook is INCORRECT

10. $\frac{dP}{dt} = 3t + 1$

$\Rightarrow dP = (3t+1)\,dt$

$\Rightarrow \int dP = \int (3t+1)\,dt$

$\Rightarrow P = \frac{3t^2}{2} + t + C$

$P(0) = 0 \Rightarrow 0 = \frac{3\,0^2}{2} + 0 + C$

$\Rightarrow C = 0$

$\therefore P(t) = \frac{3t^2}{2} + t$

12. $\frac{dy}{dx} = 2(1-y)$

$\Rightarrow \frac{1}{(1-y)}\,dy = 2dx$

$\Rightarrow \int \frac{1}{1-y}\,dy = \int 2dx$

$\Rightarrow -\ln|1-y| = 2x + C$

$\therefore \ln|1-y| = -2x - C$

Exponentiating both sides,

$\therefore |1-y| = e^{(-2x-C)}$

$\therefore |1-y| = e^{-2x} \cdot e^{-C}$

$\therefore 1-y = \pm e^{-C} e^{-2x}$

$\therefore 1-y = C_1 e^{-2x}$ (Renaming $\pm e^{-C} = C_1$)

$\therefore y = 1 - C_1 e^{-2x}$

We know $y_0 = 2$ when $x = 0$

$\therefore 2 = 1 - C_1 e^{-2(0)} = 1 - C_1$

$\Rightarrow C_1 = -1$

$\therefore y = 1 - (-1)e^{-2x} = \boxed{1 + e^{-2x}}$

16. $\frac{dN}{dt} = 5 - N$

$\Rightarrow \quad \frac{1}{5-N}\, dN = dt$

$\int \frac{dN}{5-N} = \int dt$

$\Rightarrow \quad -\ln|5-N| = t + C$

$\Rightarrow \quad \ln|5-N| = -t - C$

$\Rightarrow \quad |5-N| = e^{(-t-c)}$ [exponentiating both sides]

$\Rightarrow \quad 5 - N = \pm e^{-c} \cdot e^{-t}$

$\Rightarrow 5 - N = C_1 e^{-t}$ (Renaming $\pm e^{-c} = C_1$)

$\Rightarrow \quad N = 5 - C_1 e^{-t}$

$N(2) = 3 \quad \Rightarrow \quad 3 = 5 - C_1 e^{-2}$

$\Rightarrow \quad C_1 e^{-2} = 2$

$\Rightarrow \quad C_1 = 2e^{2}$

$\therefore \quad N(t) = 5 - 2e^{2} e^{-t} = \underline{5 - 2e^{2-t}}$

20. a) $\frac{dW}{dt} = -\lambda W$

$\Rightarrow \frac{1}{W} dW = -\lambda dt$

$\Rightarrow \int \frac{1}{W} dW = \int -\lambda dt$

$\Rightarrow \ln|W| = -\lambda t + c$

$\Rightarrow |W| = e^{-\lambda t + c} = e^{c} e^{-\lambda t}$

$\Rightarrow W = \pm e^{c} e^{-\lambda t}$

$= C_1 e^{-\lambda t}$ (renaming $\pm e^{c} = C_1$)

$W(0) = W_0$

$\Rightarrow W_0 = C_1 e^{-\lambda(0)} = C_1$

$\therefore W(t) = W_0 e^{-\lambda t}$

b) $W(0) = 123$, $W(5) = 20$.

$\therefore W_0 e^{-\lambda(0)} = 123 \Rightarrow W_0 = 123.$

$W(5) = 20 \Rightarrow W_0 e^{-\lambda(5)} = 20$

$\Rightarrow 123 e^{-5\lambda} = 20$

$$123e^{-5\lambda} = 20$$

$$\Rightarrow e^{-5\lambda} = \frac{20}{123}$$

Take ~~[illegible]~~ ln both sides.

$$\ln(e^{-5\lambda}) = \ln\left(\frac{20}{123}\right)$$

$$\Rightarrow -5\lambda = \ln\left(\frac{20}{123}\right)$$

$$\Rightarrow \lambda = \boxed{-\frac{1}{5}\ln\left(\frac{20}{123}\right)}$$

Half-life means the time it takes to decay to half of the original amount.

$\therefore$ since we start with $W(0) = 123$, we want time t such that

$$W(t) = \frac{123}{2}$$

$$\therefore \frac{123}{2} = 123e^{-\lambda t}$$

We need to solve for t.

$$\Rightarrow \quad \frac{123}{2} = 123e^{-\lambda t}$$

$$\Rightarrow \quad \frac{1}{2} = e^{-\lambda t}$$ (Dividing through by 123)

Take ln both sides

$$\Rightarrow \ln\left(\frac{1}{2}\right) = \ln\left(e^{-\lambda t}\right) = -\lambda t$$

$$\Rightarrow \quad t = -\frac{1}{\lambda}\ln\left(\frac{1}{2}\right)$$

We computed $\lambda = -\frac{5}{2}\ln\frac{20}{123}$.

$$\therefore \quad t = \frac{-\ln\left(\frac{1}{2}\right)}{-\frac{5}{2}\ln\left(\frac{20}{123}\right)}$$

$$= \boxed{\frac{2}{5}\,\frac{\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{20}{123}\right)}}$$

21. a) $$\frac{dN}{dt} = \frac{1}{100} N^2$$

$$\Rightarrow \frac{1}{N^2}\, dN = \frac{1}{100}\, dt$$

$$\Rightarrow \int \frac{1}{N^2}\, dN = \int \frac{1}{100}\, dt$$

$$\Rightarrow -\frac{1}{N} = \frac{t}{100} + C$$

$$\therefore \frac{1}{N} = -\frac{t}{100} - C$$

$$\therefore N = \frac{1}{-\frac{t}{100} - C}$$

$N(0) = 10$

$$\therefore 10 = \frac{1}{-\frac{0}{100} - C} = -\frac{1}{C}$$

$$\Rightarrow C = -\frac{1}{10}$$

$$\therefore N(t) = \frac{1}{-\frac{t}{100} - \left(-\frac{1}{10}\right)}$$

$$\boxed{N(t) = \frac{1}{\frac{-t+10}{100}} = \frac{100}{10-t}}$$

22. a) $\frac{dL}{dt} = k(34-L)$

$$\frac{dL}{34-L} = k\,dt$$

$$\int \frac{dL}{34-L} = \int k\,dt$$

$$\Rightarrow -\ln|34-L| = kt + C$$

$$\Rightarrow \ln|34-L| = -kt - C$$

Exponentiating both sides,

$$|34-L| = e^{-kt-C} = e^{-C}e^{-kt}$$

$$\therefore\ 34-L = \pm e^{-C}e^{-kt}$$

$$= C_1 e^{-kt}$$

$$\therefore\ L = 34 - C_1 e^{-kt}$$

$L(0) = 2 \Rightarrow 2 = 34 - C_1 e^{-k(0)} = 34 - C_1$

$\therefore \; C_1 = 34 - 2 = 32 \quad \therefore \boxed{L(t) = 34 - 32e^{-kt}}$

b) $L(4) = 10$

$\Rightarrow \quad 10 = 34 - 32e^{-k(4)}$

$\Rightarrow \quad 32e^{-4k} = 34 - 10 = 24$

$\Rightarrow \quad e^{-4k} = \dfrac{24}{32}$

$\Rightarrow \quad \ln(e^{-4k}) = \ln\left(\dfrac{24}{32}\right)$

$\Rightarrow \quad -4k = \ln\left(\dfrac{24}{32}\right)$

$\Rightarrow \quad k = -\dfrac{1}{4}\ln\left(\dfrac{24}{32}\right)$

NO NEED TO GRAPH.

c) when $t = 10$,

$$L(10) = 34 - 32e^{-k(10)}$$

$$= 34 - 32e^{-\left(-\frac{1}{4}\ln\left(\frac{24}{32}\right)\right)10}$$

$$= 34 - 32e^{\left(\frac{10}{4}\ln\left(\frac{24}{32}\right)\right)}$$

d) $\lim_{t \to \infty} L(t) = \lim_{t \to \infty} 34 - 32 e^{-kt}$

goes to 0.

$= \boxed{34}$

<u>29.</u> $\frac{dy}{dx} = 2y(3-y)$

$\frac{dy}{y(3-y)} = 2dx$

$\Rightarrow \int \frac{dy}{y(3-y)} = \int 2dx$

First integrate $\int \frac{dy}{y(3-y)}$

Using partial fractions

$\frac{1}{y(3-y)} = \frac{A}{y} + \frac{B}{3-y}$

$= \frac{A(3-y) + By}{y(3-y)}$

$= \frac{(B-A)y + 3A}{y(3-y)}$

$\therefore \quad B - A = 0 \quad$ and $\quad 3A = 1$

$\Rightarrow \quad A = B \quad$ and $\quad A = \frac{1}{3}$

$\therefore \quad B = \frac{1}{3}$

$$\therefore \int \frac{1}{y(3-y)}\,dy = \int \frac{\left(\frac{1}{3}\right)}{y} + \frac{\left(\frac{1}{3}\right)}{3-y}\,dy$$

$$= \frac{1}{3}\ln|y| - \frac{1}{3}\ln|3-y|$$

$$\therefore \int \frac{dy}{y(3-y)} = \int 2\,dx$$

becomes:

$$\frac{1}{3}\left(\ln|y| - \ln|3-y|\right) = 2x + C$$

$$\Rightarrow \quad \ln\frac{|y|}{|3-y|} = 3(2x + c)$$
$$= 6x + 3C$$

Exponentiating both sides,

$$\frac{|y|}{|3-y|} = e^{(6x+3C)}$$

$$\Rightarrow \quad \frac{y}{3-y} = \pm e^{3c} \cdot e^{6x}$$

$$\therefore \quad \frac{y}{3-y} = C_1 e^{6x} \quad \left(\text{Renaming } \pm e^{3c} = C_1\right)$$

$$\therefore \quad y = (3-y)(C_1 e^{6x})$$

$$\Rightarrow \quad y = 3C_1 e^{6x} - yC_1 e^{6x}$$

$$\Rightarrow \quad y + yC_1 e^{6x} = 3C_1 e^{6x}$$

$$\Rightarrow \quad y(1 + C_1 e^{6x}) = 3C_1 e^{6x}$$

$$\Rightarrow \quad y = \frac{3C_1 e^{6x}}{1 + C_1 e^{6x}}$$

Since $y = 5$ when $x = 1$

we get $$5 = \frac{3C_1 e^{6(1)}}{1 + C_1 e^{6(1)}}$$

$$= \frac{3C_1 e^{6}}{1 + C_1 e^{6}}$$

$$\Rightarrow \quad 5(1+C_1e^6) = 3C_1e^6$$

$$\Rightarrow \quad 5 + 5C_1e^6 = 3C_1e^6$$

$$\Rightarrow \quad 5 = -2C_1e^6$$

$$\Rightarrow \quad C_1 = \frac{5}{-2e^6}$$

$$\therefore \quad y = \frac{3\left(\frac{5}{-2e^6}\right)e^{6x}}{1 + \left(\frac{5}{-2e^6}\right)e^{6x}}$$

30.

$$\frac{dy}{dt} = \frac{1}{2}y^2 - 2y = y\left(\frac{1}{2}y - 4\right)$$

$$= \frac{1}{2}y(y-8)$$

$$\Rightarrow \quad \frac{dy}{y(y-8)} = \frac{1}{2}dt$$

$$\Rightarrow \quad \int \frac{dy}{y(y-8)} = \int \frac{1}{2}dt$$

First use partial fractions to get.

$$\frac{1}{y(y-8)} = \frac{A}{y} + \frac{B}{y-8}$$

$$= \frac{A(y-8) + By}{y(y-8)}$$

$$= \frac{(A+B)y + (-8A)}{y(y-8)}$$

$\therefore \quad A+B=0 \quad , \quad -8A = 1$

$\Rightarrow \quad A = -\frac{1}{8} \quad , \quad B = \frac{1}{8}$

$$\therefore \int \frac{1}{y(y-8)}\,dy = \int \frac{(-\frac{1}{8})}{y} + \frac{(\frac{1}{8})}{y-8}\,dy$$

$$= -\frac{1}{8}\ln|y| + \frac{1}{8}\ln|y-8|$$

$$= -\frac{1}{8}\left(\ln|y| - \ln|y-8|\right)$$

$$= -\frac{1}{8}\ln\left(\frac{|y|}{|y-8|}\right)$$

$$\therefore \int \frac{dy}{y(y-8)} = \int \frac{1}{2}\,dt \quad \text{becomes:}$$

$$-\frac{1}{8}\ln\left(\frac{|y|}{|y-8|}\right) = \frac{1}{2}t + C$$

$$\Rightarrow \ln\left(\frac{|y|}{|y-8|}\right) = -4t - 8C$$

Exponentiating both sides:

$$\frac{|y|}{|y-8|} = e^{-4t-8C} = e^{-8C}e^{-4t}$$

$$\therefore \frac{y}{y-8} = \pm e^{-8C}e^{-4t}$$

$$= C_1 e^{-4t}$$

$$\Rightarrow y = (y-8)\,C_1 e^{-4t}$$

$$\Rightarrow y = y\,(C_1 e^{-4t}) - 8C_1 e^{-4t}$$

$$\Rightarrow y(1 - C_1 e^{-4t}) = -8C_1 e^{-4t}$$

$$y = \frac{-8C_1 e^{-4t}}{1 - C_1 e^{-4t}}$$

When $t = 0$, $y = -3$.

$$\therefore \quad -3 = \frac{-8C_1 e^{-4(0)}}{1 - C_1 e^{-4(0)}} = \frac{-8C_1}{1 - C_1}$$

$$\therefore \quad 3(1 - C_1) = 8C_1$$

$$\Rightarrow \quad C_1 = \frac{3}{11}$$

$$\therefore \quad \boxed{y = \frac{-8\left(\frac{3}{11}\right)e^{-4t}}{1 - \left(\frac{3}{11}\right)e^{-4t}}}$$

36. $\frac{dy}{dx} = y^2+4$

$\Rightarrow \frac{dy}{y^2+4} = dx$

$\Rightarrow \int \frac{dy}{y^2+4} = \int dx$

$\Rightarrow \int \frac{1}{4\left(\left(\frac{y}{2}\right)^2+1\right)} dy = \int dx$

$\Rightarrow$ substitute $u = \frac{y}{2}$ $\Rightarrow du = \frac{dy}{2}$

$\Rightarrow dy = 2du$

$\therefore \int \frac{1}{4\left(\left(\frac{y}{2}\right)^2+1\right)} dy = \int \frac{2du}{4(u^2+1)}$

$= \frac{1}{2}\tan^{-1}u$

$= \frac{1}{2}\tan^{-1}\frac{y}{2}$

$\therefore \int \frac{1}{y^2+4} dy = \int dx$ becomes :

$\frac{1}{2}\tan^{-1}\frac{y}{2} = x + C$

$$\therefore \quad \tan^{-1}\frac{y}{2} = 2x + 2C$$

$$\Rightarrow \quad \frac{y}{2} = \tan(x + 2C)$$

$$\Rightarrow \quad y = 2\tan(x+2C).$$

This passes through $(0,2)$.

$$\therefore \quad y = 2 \quad \text{when} \quad x = 0.$$

$$\Rightarrow \quad 2 = 2\tan(0+2C)$$

$$\Rightarrow \quad 1 = \tan(2C)$$

$$\Rightarrow \quad \tan^{-1} 1 = 2C$$

$$\Rightarrow \quad C = \frac{1}{2}\tan^{-1} 1 = \frac{1}{2}\left(\frac{\pi}{4}\right) = \frac{\pi}{8}$$

$$\therefore \quad y = 2\tan\left(x + 2\left(\frac{\pi}{8}\right)\right) = \boxed{2\tan\left(x + \frac{\pi}{4}\right)}$$

42. a)

$$\Rightarrow \quad N(t) = \frac{K}{1+\left(\frac{K}{N_0}-1\right)e^{-rt}}$$

$$\Rightarrow \quad N(t)\left(1+\left(\frac{K}{N_0}-1\right)e^{-rt}\right) = K$$

$$\Rightarrow \quad 1+\left(\frac{K}{N_0}-1\right)e^{-rt} = \frac{K}{N(t)}$$

$$\Rightarrow \quad \left(\frac{K}{N_0}-1\right)e^{-rt} = \frac{K}{N(t)} - 1 = \frac{K-N(t)}{N(t)}$$

$$\therefore \quad e^{-rt} = \frac{\left(\frac{K-N(t)}{N(t)}\right)}{\left(\frac{K}{N_0}-1\right)}$$

$$= \frac{\left(\frac{K-N(t)}{N(t)}\right)}{\left(\frac{K-N_0}{N_0}\right)}$$

Take ln both sides.

$$\ln\left(e^{-rt}\right) = \ln\left(\frac{\left(\frac{K-N(t)}{N(t)}\right)}{\left(\frac{K-N_0}{N_0}\right)}\right)$$

Using $\ln\left(\frac{A}{B}\right) = \ln A - \ln B$:

$$\Rightarrow -rt = \ln\left(\frac{K-N(t)}{N(t)}\right) - \ln\left(\frac{K-N_0}{N_0}\right)$$

$$\Rightarrow rt = -\ln\left(\frac{K-N(t)}{N(t)}\right) + \ln\left(\frac{K-N_0}{N_0}\right)$$

Using $-\ln A = \ln\left(\frac{1}{A}\right)$:

$$\Rightarrow rt = \ln\left(\frac{N(t)}{K-N(t)}\right) + \ln\left(\frac{K-N_0}{N_0}\right)$$

$$\Rightarrow r = \frac{1}{t}\ln\left(\frac{N(t)}{K-N(t)}\right) + \frac{1}{t}\ln\left(\frac{K-N_0}{N_0}\right)$$

Part b is CANCELED

45. $\frac{dy}{dx} = \frac{x+1}{y}$

$\Rightarrow y\,dy = (x+1)\,dx$

$\Rightarrow \int y\,dy = \int (x+1)\,dx$

$\Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + x + C$

$y = 2$ if $x = 0$

$\therefore \frac{2^2}{2} = \frac{0^2}{2} + 0 + C$

$\Rightarrow C = 2$

$\therefore \frac{y^2}{2} = \frac{x^2}{2} + x + 2$

$\Rightarrow y^2 = x^2 + 2x + 4$

$\Rightarrow y = \sqrt{x^2 + 2x + 4}$.

47.

$$\frac{dy}{dx} = (y+1)e^{-x}$$

$$\int \frac{1}{y+1}\,dy = \int e^{-x}\,dx$$

$$\ln|y+1| = -e^{-x} + C$$

$$\Rightarrow \quad |y+1| = e^{(-e^{-x}+C)}$$

$$= e^{C} e^{-e^{-x}}$$

$$y+1 = \pm e^{C} e^{-e^{-x}}$$

$$= C_1 e^{-e^{-x}}$$

$$y = C_1 e^{-e^{-x}} - 1$$

$y = 2$ if $x = 0$.

$$\Rightarrow \quad 2 = C_1 e^{-e^{-0}} - 1$$

$$\Rightarrow \quad 2 = C_1 e^{-1} - 1 \quad \Rightarrow \quad Ce^{-1} = 3$$

$$\Rightarrow \quad C = \boxed{3e}$$

$$\therefore \quad y = (3e)e^{-e^{-x}} - 1 = \underline{3e^{(1-e^{-x})} - 1}$$

48.

$$\frac{dy}{dx} = x^2 y^2$$

$$\Rightarrow \int \frac{1}{y^2}\,dy = \int x^2\,dx$$

$$\Rightarrow -\frac{1}{y} = \frac{x^3}{3} + C$$

$$\Rightarrow y = \frac{-1}{\frac{x^3}{3} + C}$$

$y = 1$ & $x = 1$

$$\Rightarrow 1 = \frac{-1}{1^3/3 + C} = \frac{-1}{1/3 + C}$$

$$\Rightarrow \frac{1}{3} + C = -1$$

$$\Rightarrow C = -\frac{4}{3}$$

$$\therefore y = \frac{-1}{\frac{x^3}{3} - 4/3} = \frac{3}{4 - x^3}$$

49.

$$\frac{dy}{dx} = \frac{y+1}{x-1}$$

$$\Rightarrow \int \frac{1}{y+1}\,dy = \int \frac{1}{x-1}\,dx$$

$$\Rightarrow \ln|y+1| = \ln|x-1| + C$$

Exponentiating both sides,

$$e^{\ln|y+1|} = e^{(\ln|x-1| + C)}$$

$$|y+1| = e^{C} \cdot e^{\ln|x-1|}$$

$$|y+1| = e^{C}|x-1|$$

$$\Rightarrow y+1 = \pm e^{C}(x-1)$$

$$= C_1(x-1)$$

$y = 5$ when $x = 2$.

$$\therefore \quad 5+1 = C_1(2-1) = C_1$$

$$\Rightarrow C_1 = 6.$$

$$\therefore \quad y+1 = 6(x-1)$$

$$\Rightarrow \boxed{y = 6x - 7}$$