HW 12 Solutions

MAT 17B

6. $f(x) = \frac{1}{1+x}$, $n=4$, $a=0$.

$f(a) = f(0) = \frac{1}{1+0} = 1$

$f'(a) = f'(0) = -\frac{1}{(1+x)^2}\Big|_{x=0} = -1$

$f''(a) = f''(0) = \frac{2}{(1+x)^3}\Big|_{x=0} = 2$

$f^{(3)}(a) = f^{(3)}(0) = \frac{-6}{(1+x)^4}\Big|_{x=0} = -6$

$f^{(4)}(a) = f^{(4)}(0) = \frac{24}{(1+x)^5}\Big|_{x=0} = 24$

$\therefore$ Taylor Polynomial $P(x) = 1 + \frac{(-1)(x-0)}{1!} + \frac{2(x-0)^2}{2!}$

$+ \frac{(-6)(x-0)^3}{3!} + \frac{(24)(x-0)^4}{4!}$

$= \boxed{1 - x + x^2 - x^3 + x^4}$

7. $f(x) = \cos x$, $n = 5$, $a = 0$.

$f(0) = \cos(0) = 1$

$f'(0) = -\sin(0) = 0$

$f^{(2)}(0) = -\cos(0) = -1$

$f^{(3)}(0) = +\sin(0) = 0$

$f^{(4)}(0) = \cos(0) = 1$

$f^{(5)}(0) = -\sin(0) = 0$

$\therefore$ Taylor Polynomial $= 1 + \frac{0(x-0)}{1!} + \frac{(-1)(x-0)^2}{2!} + \frac{0(x-0)^3}{3!}$

$+ \frac{1(x-0)^4}{4!} + \frac{0(x-0)^5}{5!}$

$= \boxed{1 - \frac{1}{2}x^2 + \frac{1}{24}x^4}$

---

9. $f(x) = x^5$, $n = 6$, $a = 0$

$f(0) = 0^5 = 0$

$f'(0) = 5x^4\big|_{x=0} = 0$

$f^{(2)}(0) = 20x^3\big|_{x=0} = 0$

$f^{(3)}(0) = 60x^2\big|_{x=0} = 0$

$f^{(4)}(0) = 120x\big|_{x=0} = 0$

$f^{(5)}(0) = 120$

$f^{(6)}(0) = 0$

Taylor Polynomial

$$P(x) = 0 + \frac{0(x-0)}{1!} + \frac{0(x-0)^2}{2!} + \frac{0(x-0)^3}{3!} + \frac{0(x-0)^4}{4!}$$

$$+ \frac{120(x-0)^5}{5!} + \frac{0(x-0)^6}{6!}$$

$$= \frac{120x^5}{5!} = \boxed{x^5} \quad (\text{since } 5! = 120)$$

Note: If $f(x)$ is a polynomial, the Taylor polynomial of the same ~~degree~~ (or higher) degree will be $f(x)$, i.e., $P(x) = f(x)$

This is what happened in Problem 9.

---

10. $f(x) = \sqrt{1+x}$, $n = 3$, $a = 0$.

$$f(0) = \sqrt{1+0} = 1$$

$$f'(0) = \left.\frac{1}{2\sqrt{1+x}}\right|_{x=0} = \frac{1}{2}$$

$$f^{(2)}(0) = \left.-\frac{1}{4}(1+x)^{-3/2}\right|_{x=0} = -\frac{1}{4}$$

$$f^{(3)}(0) = \left.\frac{3}{8}(1+x)^{-5/2}\right|_{x=0} = \frac{3}{8}$$

$$\therefore P(x) = 1 + \frac{\left(\frac{1}{2}\right)(x-0)}{1!} + \frac{\left(-\frac{1}{4}\right)(x-0)^2}{2!} + \frac{\left(\frac{3}{8}\right)(x-0)^3}{3!}$$

$$= \boxed{1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3}$$

---

13. $f(x) = \sin x$, $n = 5$, $a = 0$.

Find value of $P(x)$ at $x = 0.1$.

$f(a) = f(0) = \sin(0) = 0$

$f'(a) = f'(0) = \cos(0) = 1$

$f^{(2)}(0) = -\sin(0) = 0$

$f^{(3)}(0) = -\cos(0) = -1$

$f^{(4)}(0) = \sin(0) = 0$

$f^{(5)}(0) = \cos(0) = 1$

$$P(x) = 0 + \frac{1(x-0)}{1!} + \frac{0(x-0)^2}{2!} + \frac{(-1)(x-0)^3}{3!} + \frac{0(x-0)^4}{4!} + \frac{1(x-0)^5}{5!}$$

$$= x - \frac{x^3}{6} + \frac{x^5}{120}$$

At $x = 0.1$

$$P(0.1) = \boxed{0.1 - \frac{(0.1)^3}{6} + \frac{(0.1)^5}{120}}$$

14. $f(x) = e^{-x}$, $n = 4$, $a = 0$ (evaluate at $x = 0.3$)

$$f(a) = f(0) = e^{-0} = 1$$

$$f'(a) = f'(0) = -e^{-x}\Big|_{x=0} = -1$$

$$f^{(2)}(0) = e^{-x}\Big|_{x=0} = 1$$

$$f^{(3)}(0) = -e^{-x}\Big|_{x=0} = -1$$

$$f^{(4)}(0) = e^{-x}\Big|_{x=0} = 1$$

$$P(x) = 1 + \frac{(-1)(x-0)}{1!} + \frac{(1)(x-0)^2}{2!} + \frac{(-1)(x-0)^3}{3!} + \frac{(1)(x-0)^4}{4!}$$

$$= 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{120}$$

At $x = 0.3$, $P(0.3) = 1 - 0.3 + \frac{(0.3)^2}{2} - \frac{(0.3)^3}{6} + \frac{(0.3)^4}{120}$

---

16. $f(x) = \ln(1+x)$, $n = 3$, $a = 0$, evaluate at $x = 0.1$.

$$f(a) = f(0) = \ln(1+0) = \ln 1 = 0$$

$$f'(a) = f'(0) = \frac{1}{1+x}\Big|_{x=0} = 1$$

$$f^{(2)}(a) = f^{(2)}(0) = -\frac{1}{(1+x)^2}\Big|_{x=0} = -1$$

$$f^{(3)}(a) = f^{(3)}(0) = \frac{2}{(1+x)^3}\Big|_{x=0} = 2$$

$$\therefore\ P(x) = 0 + \frac{(1)(x-0)}{1!} + \frac{(-1)(x-0)^2}{2!} + \frac{2(x-0)^3}{3!}$$

$$= \boxed{x - \frac{x^2}{2} + \frac{x^3}{3}}$$

At $x = 0.1$, $P(0.1) = \boxed{0.1 - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3}}$

---

19. $f(x) = \sqrt{x}$, $n = 3$, $a = 1$, evaluate at $x = 2$.

$f(a) = f(1) = \sqrt{1} = 1$

$f'(a) = f'(1) = \left.\frac{1}{2\sqrt{x}}\right|_{x=1} = \frac{1}{2}$

$f^{(2)}(a) = f^{(2)}(1) = \left.-\frac{1}{4}x^{-3/2}\right|_{x=1} = -\frac{1}{4}$

$f^{(3)}(a) = f^{(3)}(1) = \left.\frac{3}{8}x^{-5/2}\right|_{x=1} = \frac{3}{8}$

$$P(x) = 1 + \frac{(\frac{1}{2})(x-1)}{1!} + \frac{(-\frac{1}{4})(x-1)^2}{2!} + \frac{(\frac{3}{8})(x-1)^3}{3!}$$

$$= \boxed{1 + \frac{x-1}{2} - \frac{(x-1)^2}{8} + \frac{(x-1)^3}{16}}$$

At $x = 2$, $P(2) = 1 + \frac{2-1}{2} - \frac{(2-1)^2}{8} + \frac{(2-1)^3}{16} = \frac{23}{16}$

21. $f(x) = \cos x$, $a = \frac{\pi}{6}$, $n = 3$, evaluate at $x = \pi/7$.

$$f(a) = f\left(\frac{\pi}{6}\right) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$f'(a) = f'\left(\frac{\pi}{6}\right) = -\sin x\Big|_{x=\pi/6} = -\sin\pi/6 = -\frac{1}{2}$$

$$f^{(2)}(a) = f^{(2)}\left(\frac{\pi}{6}\right) = -\cos\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

$$f^{(3)}(a) = f^{(3)}(\pi/6) = +\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$P(x) = \frac{\sqrt{3}}{2} + \frac{\left(-\frac{1}{2}\right)(x-\pi/6)}{1!} + \frac{\left(-\frac{\sqrt{3}}{2}\right)(x-\pi/6)^2}{2!} + \frac{\left(\frac{1}{2}\right)(x-\pi/6)^3}{3!}$$

$$= \boxed{\frac{\sqrt{3}}{2} - \frac{(x-\pi/6)}{2} - \frac{\sqrt{3}}{4}(x-\pi/6)^2 + \frac{1}{12}(x-\pi/6)^3}$$

At $x = \pi/7$, $P(\pi/7)$

$$= \boxed{\frac{\sqrt{3}}{2} - \frac{1}{2}\left(\frac{\pi}{7}-\frac{\pi}{6}\right) - \frac{\sqrt{3}}{4}\left(\frac{\pi}{7}-\frac{\pi}{6}\right)^2 + \frac{1}{12}\left(\frac{\pi}{7}-\frac{\pi}{6}\right)^3}$$

22. $f(x) = x^{1/5}$, $\underline{a=-1}$, $n=3$ evaluate at $\underline{x=-0.9}$.

$$f(a) = f(-1) = (-1)^{1/5} = -1$$

$$f'(a) = f'(-1) = \frac{1}{5}x^{-4/5}\Big|_{x=-1} = \frac{1}{5}$$

$$f^{(2)}(a) = f^{(2)}(-1) = -\frac{4}{25}x^{-9/5}\Big|_{x=-1} = \left(-\frac{4}{25}\right)(-1)^{-9/5} = \frac{4}{25}$$

$$f^{(3)}(a) = f^{(3)}(-1) = \frac{36}{125}x^{-14/5}\Big|_{x=-1} = \frac{36}{125}$$

$$P(x) = -1 + \frac{\left(\frac{1}{5}\right)(x-(-1))}{1!} + \frac{\left(\frac{4}{25}\right)(x-(-1))^2}{2!} + \frac{\left(\frac{36}{125}\right)(x-(-1))^3}{3!}$$

$$= \boxed{-1 + \frac{(x+1)}{5} + \frac{2(x+1)^2}{25} + \frac{6}{125}(x+1)^3}$$

At $x = -0.9$.

$$P(-0.9) = -1 + \frac{(-0.9+1)}{5} + \frac{2}{25}(-0.9+1)^2 + \frac{6}{125}(-0.9+1)^3$$

$$= -1 + \frac{0.1}{5} + \frac{2(0.1)^2}{25} + \frac{6(0.1)^3}{125}$$